Suspension bridge with laminated beams*

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The longest suspension bridge in the world

With a length of 2,023 meters in its span between the two towers, the 1915 Canakkale is a tribute to the 100 years of the Turkish Republic, completed in 2023.

The most famous suspension bridge

One of the main postcards of the State of California (USA), the Golden Gate Bridge was built between 1933 and 1937 and is 2.7 km long with towers over 220 meters high.

An example of an older suspension bridge

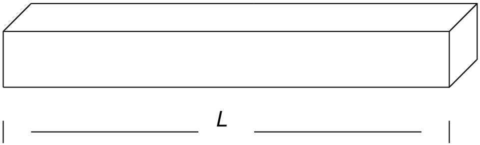
The Inca Bridge is one of the best examples of ingenious architecture of the Inca Empire (1438-1572) at Machu Picchu.

Timoshenko theory

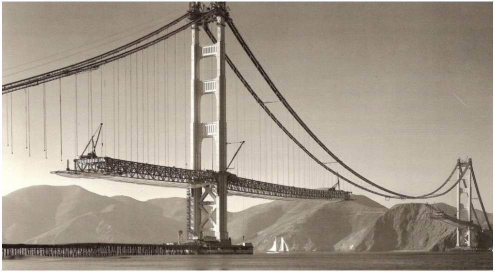
The Timoshenko's beam is given by the coupled system

$$
\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) &= 0,
\end{aligned}
$$

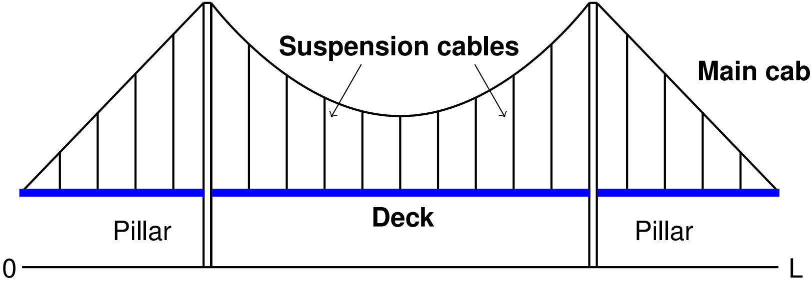
where $\varphi = \varphi(x, t)$ is the displacement of the cross-section on the point $x \in (0, L)$, $\psi = \psi(x, t)$ is the rotation angle of the cross-section, being $\rho_1 = \rho A, \rho_2 = \rho I, k = kGA, b = EI$, where ρ, A and I are, respectively, mass density, cross-section area, and moment of inertia.



Suspension bridge



Suspension bridge in Timoshenko theory



$$u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) = 0,$$
$$\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \lambda(\varphi - u) = 0,$$
$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0,$$

where

- $u = u(x,t)$ is the vertical displacement of the main cable.
- $\varphi = \varphi(x,t)$ is the displacement of the cross-section on the point $x \in (0, L)$ on the deck.
- $\psi = \psi(x,t)$ the rotation angle of the cross-section.

For suspension bridge with the deck given by Timoshenko theory we cite the works:

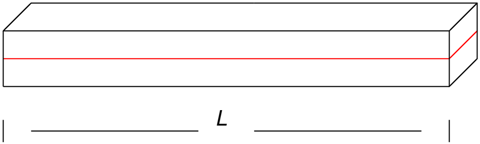
- (2020) Analysis of a thermoelastic Timoshenko beam model.
  M.G. Naso et al. *Acta Mechanica*
- (2023) Thermal Timoshenko beam system with suspenders and Kelvin-Voigt damping.
  S. E. Mukiawa et al. *Front. Appl. Math.*
- (2023) Suspension bridge with internal damping.
  C. Raposo, L. Correia, J. Ribeiro, A. Cunha. *Acta Mechanica*

Laminated beam theory

The laminated beam system of length L, is a model proposed by S. Hansen and H. Spies (1994) for two-layered beams in which a slip $s = s(x,t)$ can occur at the interface of contact, (in red at figure), given by

$$\begin{aligned} \rho\varphi_{tt} + G(\psi - \varphi_x)_x &= 0, \\ I_\rho(3S_{tt} - \psi_{tt}) - D(3S_{xx} - \psi_{xx}) - G(\psi - \varphi_x) &= 0, \\ I_\rho 3S_{tt} - D3S_{xx} + 3G(\psi - \varphi_x) + 4\gamma_0 S + 4\delta_0 S_t &= 0. \end{aligned}$$

The positive parameters ρ, I_ρ, G, D, and γ_0, are the density, mass moment of inertia, shear stiffness, flexural rigidity, and adhesive stiffness, respectively.



Suspension bridge in laminated beam theory

We consider the action of **frictional dampings** on each component and to simplify the computations, we make the following replacements: $s(x,t) = 3S(x,t)$, $\xi = (3S - \psi)(x,t)$ $\rho_1 = \rho$, $\rho_2 = I_\rho$, $k = G$, $b = D$, $3\gamma_= 4\gamma_0$, $3\mu_4 = 4\delta_0$. Then, we introduce a suspension bridge model in laminated beams as follows,

$$u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \mu_1 u_t = 0, \tag{1}$$

$$\rho_1 \varphi_{tt} + k(s - \xi - \varphi_x)_x + \lambda(\varphi - u) + \mu_2 \varphi_t = 0, \tag{2}$$

$$\rho_2 \xi_{tt} - b\xi_{xx} - k(s - \xi - \varphi_x) + \mu_3 \xi_t = 0, \tag{3}$$

$$\rho_2 s_{tt} - b s_{xx} + 3k(s - \xi - \varphi_x) + \gamma s + \mu_4 s_t = 0. \tag{4}$$

The non-negative parameters $\mu_i > 0$, $i = 1, 2, 3, 4$, are the coefficients of the damping force, and μ_4 is called the adhesive damping.

For system (1)–(4) we take $x \in (0, L)$ and $t \geq 0$ with initial data

$$
\begin{gathered}
(u(x,0), \varphi(x,0), \xi(x,0), s(x,0)) = (u_0(x), \varphi_0(x), \xi_0(x), s_0(x)), \\
(u_t(x,0), \varphi_t(x,0), \xi_t(x,0), S_t(x,0)) = (u_1(x), \varphi_1(x), \xi_1(x), s_1(x)),
\end{gathered}
$$

and Dirichlet boundary conditions

$$
\begin{aligned}
u(0,t) = \varphi(0,t) = \xi(0,t) = s(0,t) = 0, \\
u(L,t) = \varphi(L,t) = \xi(L,t) = s(L,t) = 0.
\end{aligned}
$$

Plan of the work

This work introduces a suspension bridge system where laminated beams model the deck under the action of frictional damping. The goals are:

- well-posedness.
- exponential stability.

Well-posedness

We introduce the vector function $U = (u, w, \varphi, \phi, \xi, \eta, s, z)^T$, where $\xi = s - \psi$, $w = u_t$, $\phi = \varphi_t$, $\eta = \xi_t$ and $z = s_t$.
The system (1)-(4) can be written as

$$\begin{cases} U_t - \mathcal{A}U = 0, \\ U(x,0) = U_0(x), \end{cases} \tag{5}$$

where the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ w \\ \varphi \\ \phi \\ \xi \\ \eta \\ s \\ z \end{pmatrix} = \begin{pmatrix} w \\ -[-\alpha u_{xx} - \lambda(\varphi - u) + \mu_1 w] \\ \phi \\ -\dfrac{1}{\rho_1}[k(s - \xi - \varphi_x)_x + \lambda(\varphi - u) + \mu_2\phi] \\ \eta \\ -\dfrac{1}{\rho_2}[-b\xi_{xx} - k(s - \xi - \varphi_x) + \mu_3\eta] \\ z \\ -\dfrac{1}{\rho_2}[-bs_{xx} + k(s - \xi - \varphi_x) + \gamma s + \mu_4 z] \end{pmatrix}.$$

Energy space

The energy space is given by

$$\mathcal{H} = [H_0^1(0,L) \times L^2(0,L)]^4$$

and domain of the operator $\mathcal{A}$ is

$$D(\mathcal{A}) = [H_0^1(0,L) \cap H^2(0,L) \times H_0^1(0,L)]^4.$$

In $\mathcal{H}$ we consider the following inner product

$$\begin{aligned}
\langle U, \tilde{U} \rangle_{\mathcal{H}} = {} & (w, \bar{\tilde{w}}) + \alpha(u_x, \bar{\tilde{u}}_x) \\
& + \lambda(\phi - u, \bar{\tilde{\phi}} - \bar{\tilde{u}}) \\
& + \rho_1(\phi, \bar{\tilde{\phi}}) + \rho_2(\eta, \bar{\tilde{\eta}}) + \rho_2(z, \bar{\tilde{z}}) \\
& + k(s - \xi - \varphi_x, \bar{s} - \bar{\tilde{\xi}} - \bar{\tilde{\varphi}}_x) \\
& + b(\xi_x, \bar{\tilde{\xi}}_x) + b(s_x, \bar{\tilde{s}}_x) + \gamma(z, \bar{\tilde{s}}).
\end{aligned}$$

Clear $D(\mathcal{A})$ is dense in $\mathcal{H}$, and $\mathcal{H}$ is a Hilbert space with norm

$$\|U\|_{\mathcal{H}}^2 = \langle U, U \rangle_{\mathcal{H}}.$$

Proposition

The operator $\mathcal{A}$ is dissipativo on $\mathcal{H}$, that is,
$\mathfrak{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\mu_1\|w\|^2 - \mu_2\|\phi\|^2 - \mu_3\|\eta\|^2 - \mu_4\|z\|^2.$

Lemma

$0 \in \rho(\mathcal{A})$, *the resolvent set of $\mathcal{A}$.*

The well-posedness of (1)-(4) is ensured by the following theorem.

Theorem

For $U_0 \in \mathcal{H}$, there exists a unique weak solution U of (5) *satisfying*
$U \in C^0((0,\infty);\mathcal{H})$. *Moreover, if $U_0 \in D(\mathcal{A})$, then*
$U \in C^0((0,\infty); D(\mathcal{A})) \cap C^1((0,\infty);\mathcal{H})$.

Proof As $D(\mathcal{A})$ is dense in $\mathcal{H}$, $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$. As consequence of the Lumer-Phillips theorem we have that $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$. From semigroup theory $U(t) = e^{t\mathcal{A}}U_0$ is the unique solution of (5) in theorem's conditions.

First, we prove

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

... and by using

Theorem (Gagliardo-Niremberg)

Let j and m be integers satisfying $0 \leq j < m$, and let $1 \leq q, r \leq \infty$, and $p \in \mathbb{R}$, $\dfrac{j}{m} \leq a \leq 1$ such that

$$\frac{1}{p} - \frac{j}{n} = a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

Then, for any $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, there are two positive constants C_1, C_2 such that

$$|D^j u|_{L^p(\Omega)} \leq C_1 |D^m u|^a_{L^r(\Omega)} |u|^{1-a}_{L^q(\Omega)} + C_2 |u|_{L^q(\Omega)}.$$

... finally, we prove the main result,

Theorem

The semigroup $S(t) = e^{\mathcal{A}t},\ t \geq 0$, generated by $\mathcal{A}$ is exponentially stable.

with help of the Gearhart-Huang-Prüss theorem:

Theorem (Gearhart-Huang-Prüss)

Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$. Then, $S(t)$ is exponentially stable if, only if,

$$i\mathbb{R} \subset \rho(\mathcal{A}),\ \textit{the resolvent set of } \mathcal{A}.$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty.$$

Proof of the main theorem

As $i\mathbb{R} \subset \rho(\mathcal{A})$, it remains to prove that

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty. \tag{6}$$

If (6) is not true, there exists a sequence $\beta^n \to \infty$, without loss of generality $\beta^n > 0$, a sequence of complex vectors $F^n \in \mathcal{H}$ and a corresponding sequence $U^n = (u^n, w^n, \varphi^n, \phi^n, \xi^n, \eta^n, s^n, z^n)^T \in D(\mathcal{A})$, with $\|U^n\|_{\mathcal{H}} = 1$,

$$U^n = (i\beta I - \mathcal{A})^{-1} F^n, \tag{7}$$

such that,

$$\frac{||\,(i\beta^n I - \mathcal{A})^{-1}\, F^n||_{\mathcal{H}}}{||F^n||_{\mathcal{H}}} > n, \ \ \forall\, n > n_0.$$

$$\frac{|| \left(i\beta^n I - \mathcal{A}\right)^{-1} F^n ||_{\mathcal{H}}}{||F^n||_{\mathcal{H}}} > n, \ \ \forall \, n > n_0,$$

is equivalently

$$||F_n||_{\mathcal{H}} < \frac{|| \left(i\beta^n I - \mathcal{A}\right)^{-1} F^n ||_{\mathcal{H}}}{n} = \frac{||U^n||_{\mathcal{H}}}{n} = \frac{1}{n}, \ \ \forall \, n > n_0,$$

from where follows that $F_n \to 0$ in $\mathcal{H}$.

Proof of the main theorem

Taking the inner product of $(i\beta^n I - \mathcal{A})\, U^n$ with U^n in $\mathcal{H}$ we have

$$i\beta^n ||U^n||_{\mathcal{H}} - \langle \mathcal{A}U^n, U^n\rangle_{\mathcal{H}} = \langle F^n, U^n\rangle_{\mathcal{H}}$$

Taking the real part and using

$$\mathfrak{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\mu_1\|w\|^2 - \mu_2\|\phi\|^2 - \mu_3\|\eta\|^2 - \mu_4\|z\|^2$$

we get

$$\mu_1\|w^n\|^2 + \mu_2\|\phi^n\|^2 + \mu_3\|\eta^n\|^2 + \mu_4\|z^n\|^2 = \mathfrak{Re}\,\langle F^n, U^n\rangle_{\mathcal{H}}\,.$$

Proof of the main theorem

As U^n is bounded and $F^n \to 0$ we have that

$$\mathfrak{Re}\,\langle F^n, U^n\rangle \to 0 \text{ in } \mathcal{H},$$

that is,

$$\begin{aligned} w^n &\to 0 \text{ in } L^2(0,L),\\ \phi^n &\to 0 \text{ in } L^2(0,L),\\ \eta^n &\to 0 \text{ in } L^2(0,L),\\ z^n &\to 0 \text{ in } L^2(0,L). \end{aligned}$$

Proof of the main theorem

Applying Gagliardo-Niremberg inequality we can prove that

$$\varphi^n \to 0 \text{ in } H_0^1(0,L),$$
$$\phi^n \to 0 \text{ in } L^2(0,L),$$
$$\xi^n \to 0 \text{ in } H_0^1(0,L),$$
$$\eta^n \to 0 \text{ in } L^2(0,L),$$
$$s^n \to 0 \text{ in } H_0^1(0,L),$$
$$z^n \to 0 \text{ in } L^2(0,L).$$

It follows from previous convergence that $||U^n||_{\mathcal{H}} \to 0$, which is a contradiction with $||U^n||_{\mathcal{H}} = 1$.

$\blacksquare$

Suspension bridge with von Kármám beam system,

$$\begin{cases} u_{tt} - \alpha u_{xx} - \lambda(\omega - u) = 0, \\ \omega_{tt} - b_1 \left[\left(\varphi_x + \dfrac{1}{2}\omega_x^2 \right) \omega_x \right]_x + b_2 \omega xxxx + \lambda(\omega - u) = 0, \\ \varphi_{tt} - b_1 \left[\varphi_x + \dfrac{1}{2}\omega_x^2 \right]_x = 0, \end{cases}$$

with the collaboration of

- Roseane Martins (UFBA).
- Joilson Oliveira Ribeiro (UFBA).
- Octavio Vera Villagran (Universidad of Tarapacá - Chile).

Future works

Suspension bridge in truncated Timoshenko-Ehrenfest theory,

$$
\begin{aligned}
u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) &= 0,\\
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \lambda(\varphi - u) &= 0,\\
-\rho_2 \varphi_{ttx} - b\psi_{xx} + k(\varphi_x + \psi) &= 0,
\end{aligned}
$$

with the collaboration of

- Gutemberg Miranda (UFPA).
- Isaac Elishakoff (Florida Atlantic University - USA).

Suspension bridge where the deck is given by a circular arch modeled by a Bresse system,

$$
\begin{aligned}
u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) &= 0, \\
\rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi + \ell\omega)_x - \ell\kappa_0(\omega_x - \ell\varphi) + \lambda(\varphi - u) &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + \ell\omega) &= 0, \\
\rho_1 \omega_{tt} - \kappa_0(\omega_x - \ell\varphi)_x + \ell\kappa(\varphi_x + \psi + \ell\omega) &= 0,
\end{aligned}
$$

with the collaboration of

- Sebastião Cordeiro (UFPA).

Bibliography-Laminated beam

- S. P. Timoshenko, *On the correction for shear of the differential equation for transverse vibration of prismatic bars*, Philosophical Magazine, 41 (1921), 744–746.
- S. W. Hansen, *Control and estimation of distributed parameter systems: Nonlinear Phenomena*, International Series of Numerical Analysis, 118 (1994), 143–170.
- S. W. Hansen, R. Spies, *Structural damping in a laminated beams due to interfacial slip*, Journal of Sound and Vibration, 204 (1997), 183–202.
- C. A. Raposo, *Exponential stability for a structure with interfacial slip and frictional damping*, Applied Mathematics Letters, 53 (2016), 85–91.

Bibliography-Semigroup theory

- L. Gearhart, *Spectral theory for contraction semigroups on Hilbert space*, Transactions of the American Mathematical Society, 236 (1978), 385–394.
- F. L. Huang, *Characteristic conditions for exponential stability of linear dynamical system in Hilbert spaces*, Annals of Differential Equations 1 (1985) 43–56 .
- J. Prüss, : *On the Spectrum of C_0-Semigroups*, Transactions of the American Mathematical Society, 284 (1984), 847–857.
- Z. Liu, S. Zheng: *Semigroups Associated with Dissipative Systems*, Springer, 2011.
- A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983

Bibliography-Suspension bridge

- C. A. Raposo, *Suspension bridge model with laminated beam*, Mathematica Moravica, 27(2) (2023), 77-90.
- C. A. Raposo, L. Correia, J. Ribeiro, A. Cunha *Suspension bridge with internal damping*, Acta Mechanica, (2023). https://doi.org/10.1007/s00707-023-03744-7
- S. E. Mukiawa, Y. Khan, H. Al Sulaimani, *Thermal Timoshenko beam system with suspenders and Kelvin-Voigt damping*, Frontiers in Applied Mathematics and Statistics, (2023), 1–7.
- I. Bochicchio, M. Campo, J.R. Fernández, M.G. Naso, *Analysis of a thermoelastic Timoshenko beam model*, Acta Mechanica, 231 (2020), 4111–4127.

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