Berry-Esseen estimates based on generators with applications

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Basic definitions

Let B be a Banach space with norm $||\cdot||$.

Definition

A one-parameter family $\{\mathrm{T}(t) : t \geq 0\}$ of bounded linear operators on a Banach space B is called semi-group if

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Moreover, a semigroup $\{\mathrm{T}(t) : t \geq 0\}$ on B is said to be strongly continuous if $\lim_{t\to 0} \mathrm{T}(t)f = f$ for every $f \in \mathrm{B}$;
it is said to be a contraction semigroup if $||\mathrm{T}(t)|| \leq 1$ for all $t \geq 0$.

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The (infinitesimal) generator of a semigroup $\{\mathsf{T}(t)\}$ on B is the linear operator defined by

$$\mathsf{L}f = \lim_{t \to 0} \frac{1}{t} \left[\mathsf{T}(t)f - f\right].$$

The domain $\mathcal{D}(\mathsf{L})$ of L is the subspace of all $f \in \mathsf{B}$ for which this limit exists.

Proposition

Let $\{\mathsf{T}(t)\}$ be a strongly contraction semigroup on B with generator L.

1. If $f \in \mathsf{B}$ and $t \geq 0$, then $\int_0^t \mathsf{T}(s) f \, ds \in \mathcal{D}(\mathsf{L})$ and

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Theorem[[4]Hille-Yosida]
A linear operator L on B is the generator of a strongly continuous contraction semigroup on B if and only if

1. $\mathcal{D}(\mathrm{L})$ is dense in B;
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Definition Let L be a closed linear operator on B. A subspace $\mathcal{D}$ of $\mathcal{D}(\mathrm{L})$ is said to be a core for L if the closure of the restriction of L to $\mathcal{D}$ is equal to L.

Motivation

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2. Let $\pi_n : \mathrm{B} \to \mathrm{B}_n$ the natural projection such that $\sup_n ||\pi_n||_{\mathrm{OP}} < \infty$
3. We say that a sequence $\{f_n : n \in \mathbb{N}\}$ with $f_n \in \mathrm{B}_n$ for every $n \in \mathbb{N}$, converges to some $f \in \mathrm{B}$ (and write $f_n \to f$) if $||\pi_n f - f_n|| \to 0$ as $n \to \infty$.

Theorem[[4]Trotter-Kato] For $n = 1, 2, \dots$, let $\{\mathrm{T}_n(t) : t \geq 0\}$ and $\{\mathrm{T}(t) : t \geq 0\}$ be strongly contraction semigroups on B_n and B with generators L_n and L. Let $\mathcal{D}$ be a core for L. Then, the following are equivalent

1. For each $f \in \mathrm{B}$, $\mathrm{T}_n(t)\pi_n f \to \mathrm{T}(t)f$ for all $t \geq 0$ uniformly on bounded intervals.
2. For each $f \in \mathrm{B}$, $\mathrm{T}_n(t)\pi_n f \to \mathrm{T}(t)f$ for all $t \geq 0$
3. For each $f \in \mathcal{D}$, there exists $f_n \in \mathcal{D}(\mathrm{L}_n)$ for each $n \geq 1$ such that $f_n \to f$ and $\mathrm{L}_n f_n \to \mathrm{L}f$.

Also, to have the above theorem, we must assume that for $f \in \mathcal{D}(\mathrm{L})$, $\pi_n \mathrm{T}(t) f \in \mathcal{D}(\mathrm{L}_n)$ for all $t \geq 0$ and that $\mathrm{L}_f \pi_n \mathrm{T}(\cdot) f : [0, \infty) \to \mathrm{B}_n$ is continuous.

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And the answer is, we are working in this problem, but with the conditions slightly changed

Consider $\mathrm{B}_n, \mathrm{B}, \mathrm{T}_n(t), \mathrm{T}(t), \mathrm{L}_n, \mathrm{L}$ and π_n as the same as in the previous Theorem and for each $n \in \mathbb{N}$ define $\Xi_n : \mathrm{B} \to \mathrm{B}_n$ a family of linear operators.

For easy notation, denote $A_n = \pi_n + \Xi_n$. Let us add some hyphoteses

(H1) for each $f \in \mathcal{D}(\mathrm{L})$, the mapping $\mathrm{L}_n A_n \mathrm{T}(\cdot) f : [0, \infty) \to \mathrm{B}_n$ is integrable;

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(H1) for each $f \in \mathcal{D}(\mathsf{L})$, the mapping $\mathsf{L}_n A_n \mathsf{T}(\cdot) f : [0, \infty) \to \mathsf{B}_n$ is integrable;

(H2) There exists functions $s_1(n) \downarrow 0$ and $s_2(n) \downarrow 0$ such that, for any $f \in \mathcal{D}(\mathsf{L}^2)$

$$||\pi_n \mathsf{L} f - \mathsf{T}_n(t) A_n f|| \leq s_1(n) ||\mathsf{L} f|| + s_2(n) ||\mathsf{L}^2 f||$$

(H3) There exists functions $r_1(n) \downarrow 0$ and $r_2(n) \downarrow 0$ such that, for any $f \in \mathcal{D}(\mathsf{L}^2)$

$$||\Xi_n f|| \leq r_1(n) ||f|| + r_2(n) ||\mathsf{L} f||$$

Trotter-Kato with estimates

Theorem Under the Hypothesis (H1) - (H3), for any $f \in \mathcal{D}(\mathsf{L}^2)$ and for each $t \geq 0$ in a compact interval, we have that

$$||\pi_n \mathsf{T}(t) f - \mathsf{T}(t)\pi_n f|| \leq r_1(n)||f|| + \max\{s_1(n), r_1(n), r_2(n)\}||\mathsf{L}f|| + \max\{s_2(n), r_2(n)\}||\mathsf{L}^2||$$

Remark The Theorem above was solved joint with Milton Jara when he was visiting Salvador last year.

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(S, d) a locally compact Polish space, and denote by $\mathcal{C}_0(\mathsf{S})$ the space of the functions which vanish at infinity. Let $\{f_k\}$ be a dense

sequence of functions in $\mathcal{C}_0(\mathsf{S})$, and define a metric in the space of subprobability measure over S, $d : \mathcal{M}_{\leq 1}(\mathsf{S}) \times \mathcal{M}_{\leq 1}(\mathsf{S}) \mapsto [0, \infty)$ given by

$$\mathbf{d}(\mu, \nu) = \sum_{k=0}^{\infty} \frac{1}{2^k} \left[\left| \int f_k \, d\mu - \int f_k \, d\nu \right| \wedge 1 \right] \tag{1}$$

To state the Theorem, we must add some more Hypothesis.

(G1) For each f_k of the dense family chosen to define the metric must satisfy

a) $f_k \in \mathcal{D}(\mathsf{L}^2)$

b) The norms $||f_k||$, $||\mathsf{L} f_k||$ and $||\mathsf{L}^2 f_k||$ cannot grows faster than 2^k.

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b) The norms $||f_k||$, $||\mathsf{L}f_k||$ and $||\mathsf{L}^2 f_k||$ cannot grows faster than 2^k.

(G2) The Banach space B is $\mathcal{C}_0(\mathsf{S})$ and π_n is the restriction to a subset S_n of S.

Theorem[Berry-Esseen estimates on the distance $\mathbf{d}$ and functional CLT] Assume the hyphoteses (H1) - (H3) and (G1)-(G2). Let

$\{X(t) : t \geq 0\}$ and $\{X_n(t) : t \geq 0\}$ be the Feller processes on S and S_n, respectively assumed to start at the same point. Fix some $t \in [0, T]$ and denote by μ and μ_n the probability distributions on S induced by $X(t)$ and $X_n(t)$, respectively. Then

$$\mathbf{d}(\mu, \mu_n) \lesssim \max\{s_1(n), s_2(n), r_1(n), r_2(n)\}$$

Moreover, $X_n \Rightarrow X$ in the space $\mathrm{D}_{\mathrm{S}}[0, \infty)$

Application

Let us fix some notation.
By $B_0(t)$ we denote the usual absorbed Brownian motion in $[0, \infty)$ and by Δ we denote a cemetery state.

Definition

A general Brownian motion on $[0, \infty)$ is a diffusion process W on $\mathbb{G} = \Delta \cup [0, \infty)$ such that the absorbed process $W(t \wedge T(0))$, $t \geq 0$ on $[0, \infty)$ has the same distribution as B_0 for any starting point $x \geq 0$.

The general Brownian motion is a continuous stochastic process with boundary condition at zero which can be seen as a mixture of the absorbed, reflected and killed Brownian motion

Theorem[[6]] Any general Brownian motion W on $[0, \infty)$ has generator $\mathsf{L} = \frac{1}{2}\frac{d^2}{dx^2}$ with corresponding domain

$$\mathcal{D}(\mathsf{L}) = \{f \in \mathcal{C}_0(\mathbb{G}) : af(0) - bf'(0) + \frac{1}{2}cf''(0) = 0\}$$

for some $a, b, c \geq 0$, such that $a + b + c = 1$ and $a \neq 1$

The random walk with boundary conditions depending on parameters A, B, α and β, is a Feller process on $\mathbb{G}_n = \frac{1}{n}\mathbb{Z}^+ \cup \Delta$ denoted by X_n, with semigroup $\{\mathsf{T}_n(t) : t \geq 0\}$ and generator L_n indexed in $n \in \mathbb{N}$ acting on local functions $f : \mathbb{G}_n \to [0, \infty)$ as follows

$$\mathsf{L}_n f(x) = \sum_{y;|x-y|=1} \eta^n_{x,y}[f(y) - f(x)]$$

where

$$\eta^n_{x,y} = \begin{cases} \frac{B}{n^\beta}, & \text{if } x = 0 \text{and} y = \Delta; \\ \frac{A}{n^\alpha}, & \text{if } x = 0 \text{and} y = \frac{1}{n}; \\ 1, & \text{otherwise.} \end{cases}$$

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Thank you!!!

[1, 7, 5, 2, 3, 8]