Law of Large Numbers for Random Walks in Conservative Random Environments

Encontro de Pós Graduação UFBA

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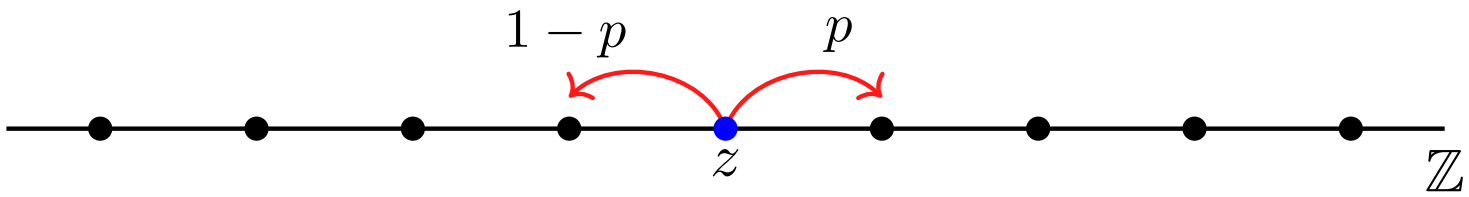
Random walks

A particle moves randomly.

At each unit of time, it jumps right with probability p and to the right with probability $1 - p$.



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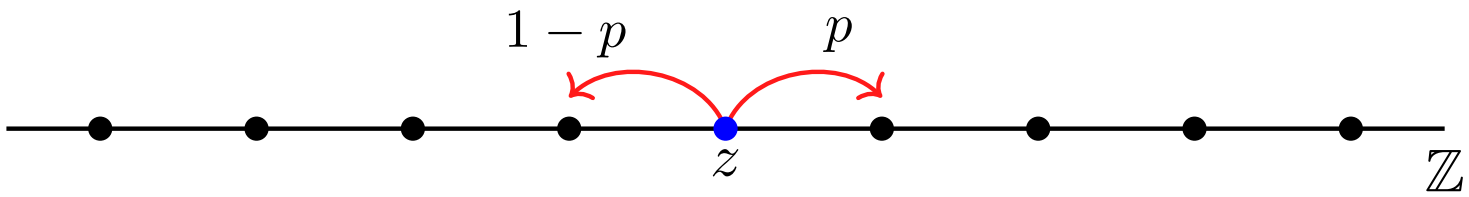
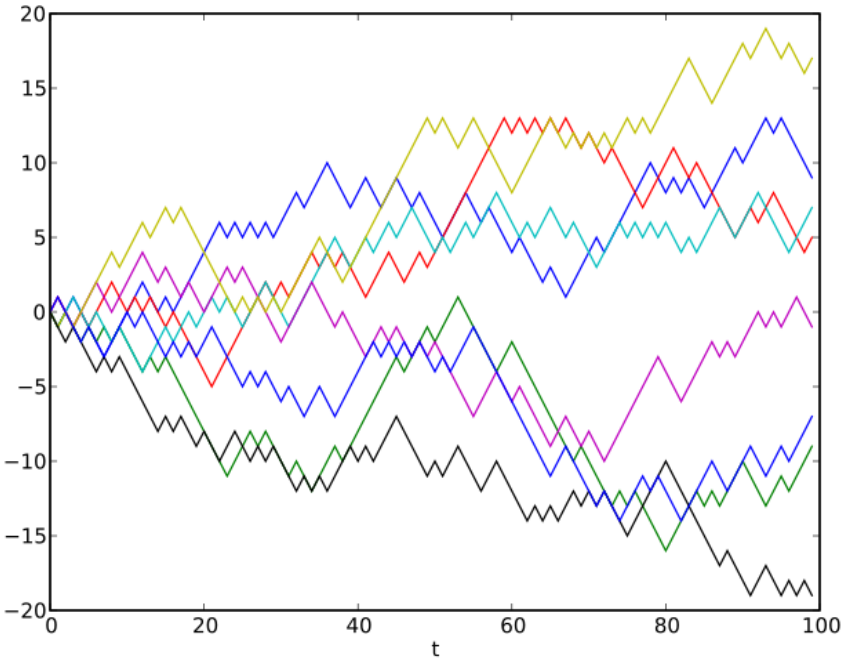


Trajectories

- $X_1, X_2, X_3, \ldots$ independent random increments.
- $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = 0) = 1 - p$.
- $S_n := X_1 + \cdots + X_n$.



Law of Large Numbers

$$\frac{S_n}{n} \to \mu, \quad (\mu = 2p - 1).$$

Central Limit Theorem

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0, 1).$$

The Brownian motion

Figure: Robert Brown (1773-1858)

Figure: Albert Einstein (1879-1955)

The Wiener Process

- $\left(\frac{S_{[nt]}-[nt]\mu}{\sigma\sqrt{[nt]}}\right)_{t\geq 0} \longrightarrow (B_t)_{t\geq 0}.$
- Continuous trajectories, $B_0 = 0$, $B_t - B_s \sim N(0, t-s)$, independent increments.

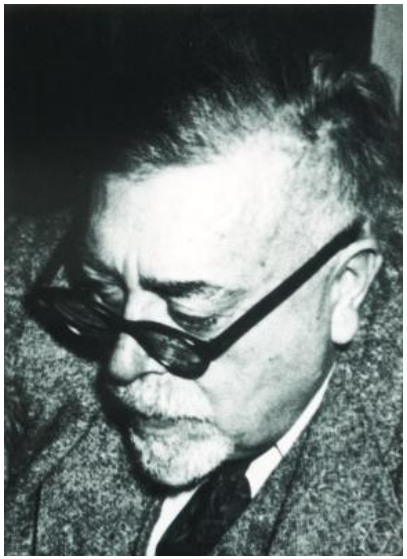
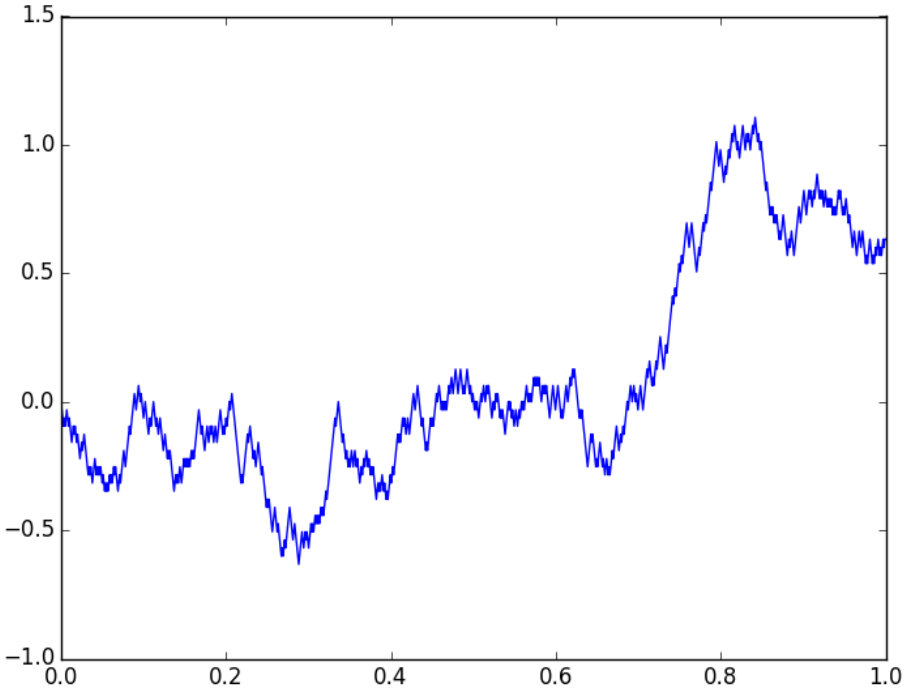


Figure: Norbert Wiener (1894 -1964)



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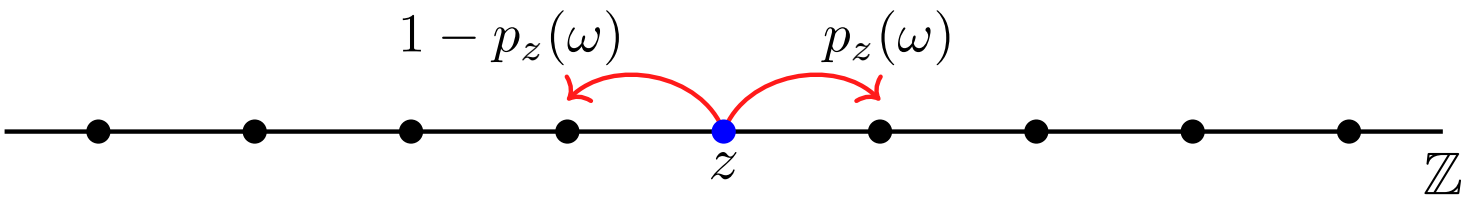

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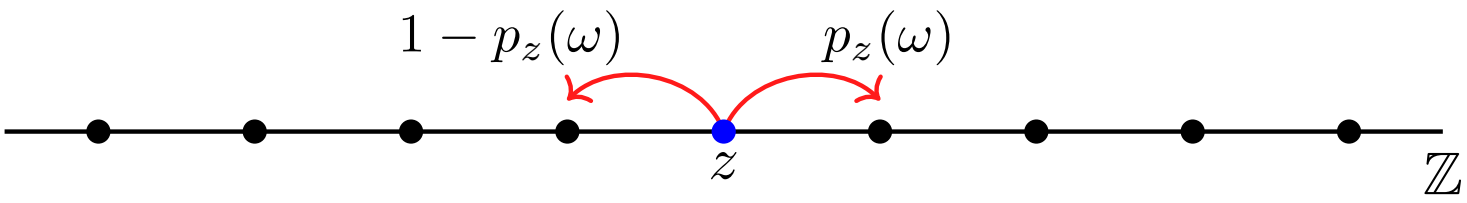

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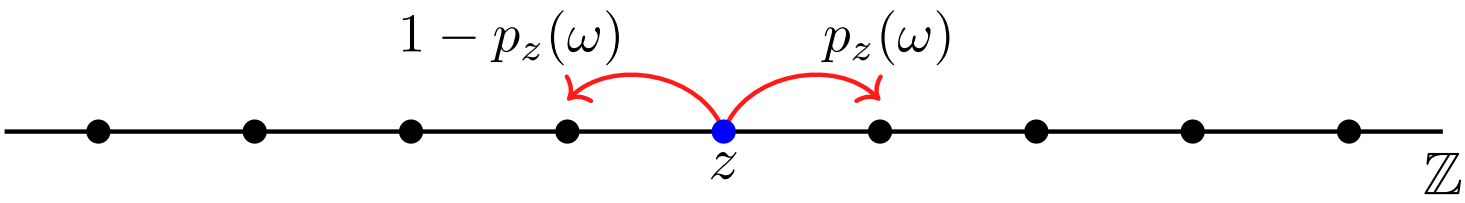

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  recurrence vs. transience, LLN, CLT, LDP, anomalous diffusion, transience with zero speed.

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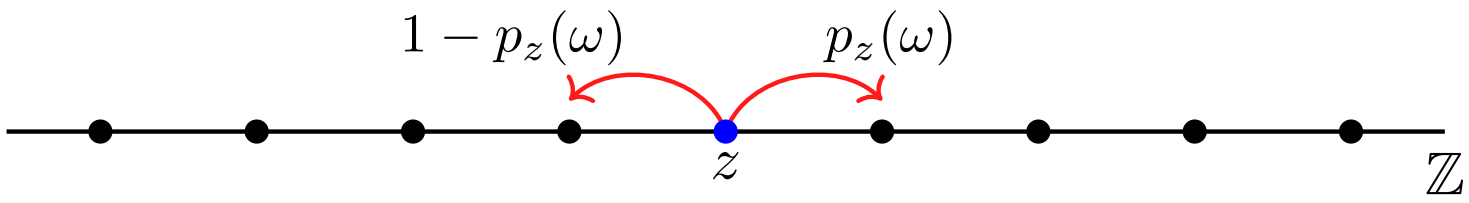

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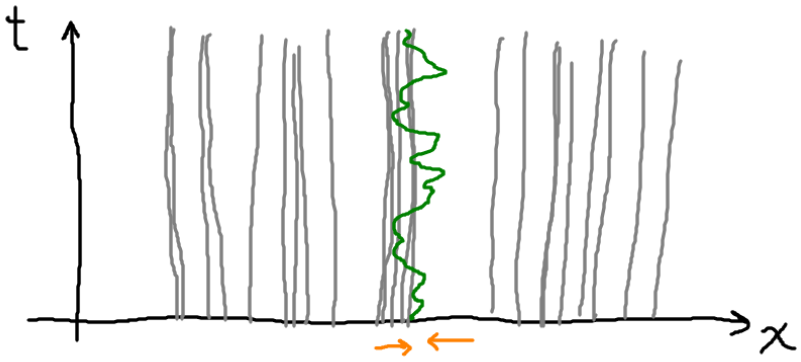
- Higher dimensions: active research.

  Berger, Comets, Drewitz, Fribergh, Ramirez, Sznitman, Zeitouni, Zerner...

Dynamic random environment



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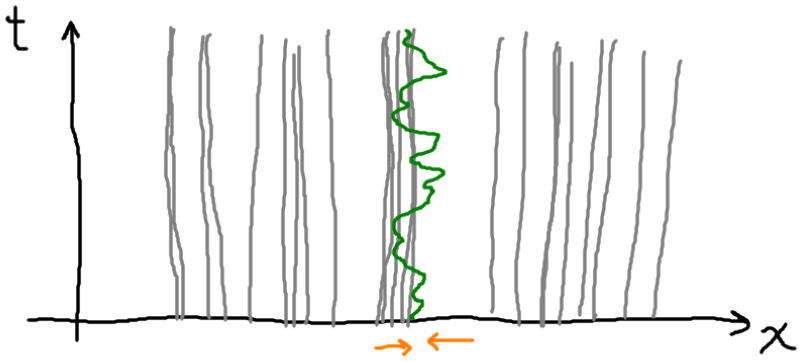

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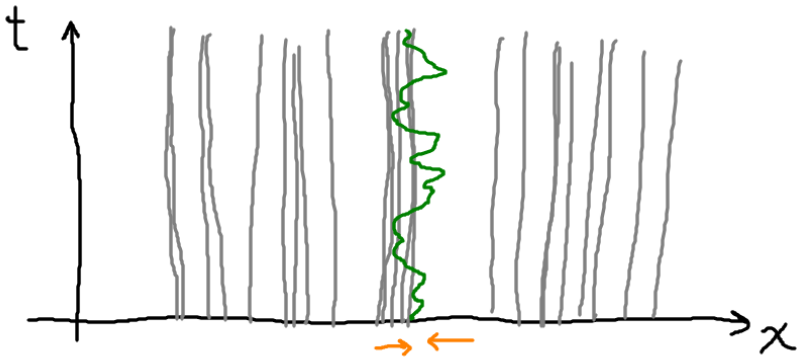

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- e.g. interacting particle systems: exclusion, contact, ind. random walks, zero-range, spin flip, ...
- Conservative systems: poor mixing, non-uniform, slow rate.
- **Question:** Are trapping effects still relevant?

Some literature

- Cone-mixing: Comets, Zeitouni (2004), Avena, den Hollander, Redig (2010).

  Remark: General, uniform-mixing.
- Supercritical contact process: dos Santos, den Hollander (2014); Mountford, Vares (2015).

  Remark: Specific features of contact process.
- Simple symmetric exclusion: Avena, dos Santos and Vollering (2011), Huveneers and Simenhaus (2015).

  Remark: Perturbative in drift of random walk or in the rate of the environment.

  Avena, Franco, Jara and Vollering (2015), Jara and Menezes (2020), H.,Teixeira, Kious (2020).
- Independent random walks: den Hollander, Kesten, Sidoravicius (2013); H., den Hollander, dos Santos, Sidoravicius, Teixeira (2015).

  Remark: Perturbative in the density of the environment.

The model

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- $\eta = (\eta_t)_{t \in \mathbb{R}_+}$ interacting particle system.

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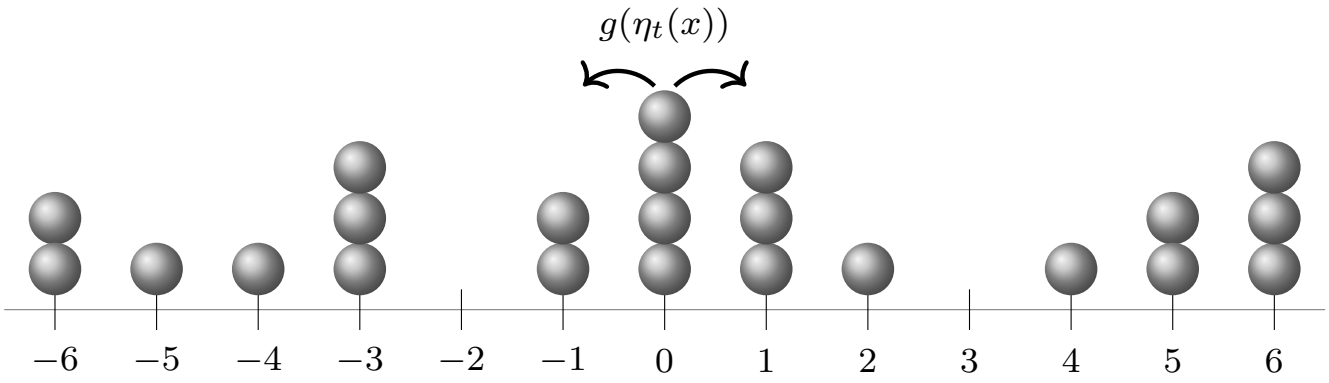

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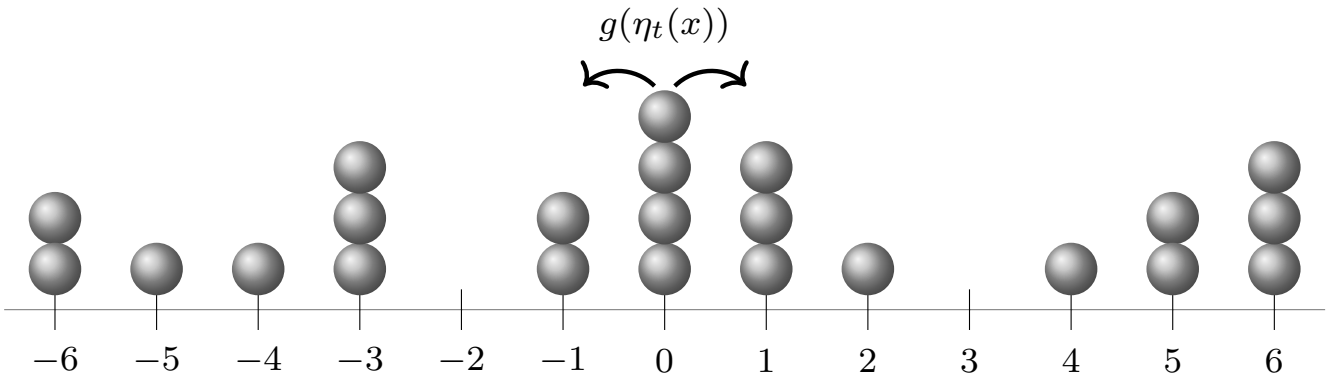
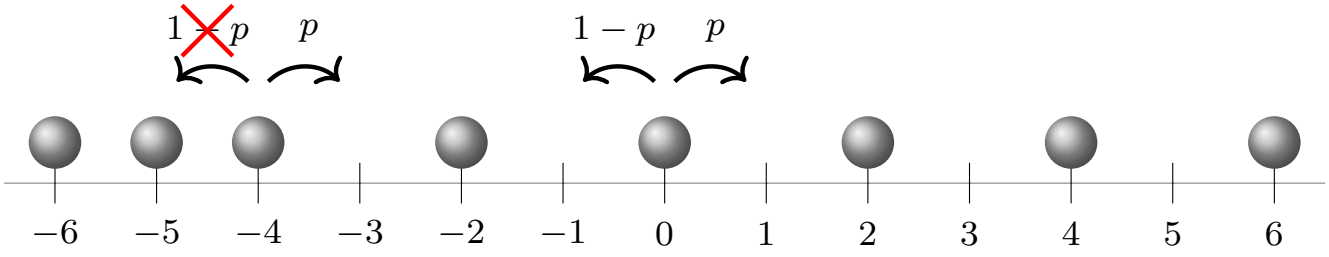


- Exclusion:



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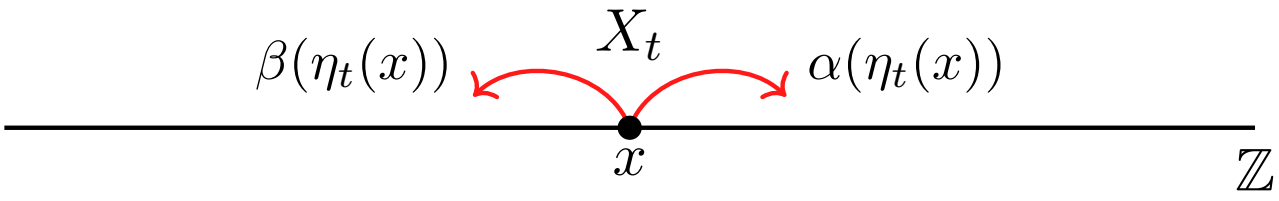


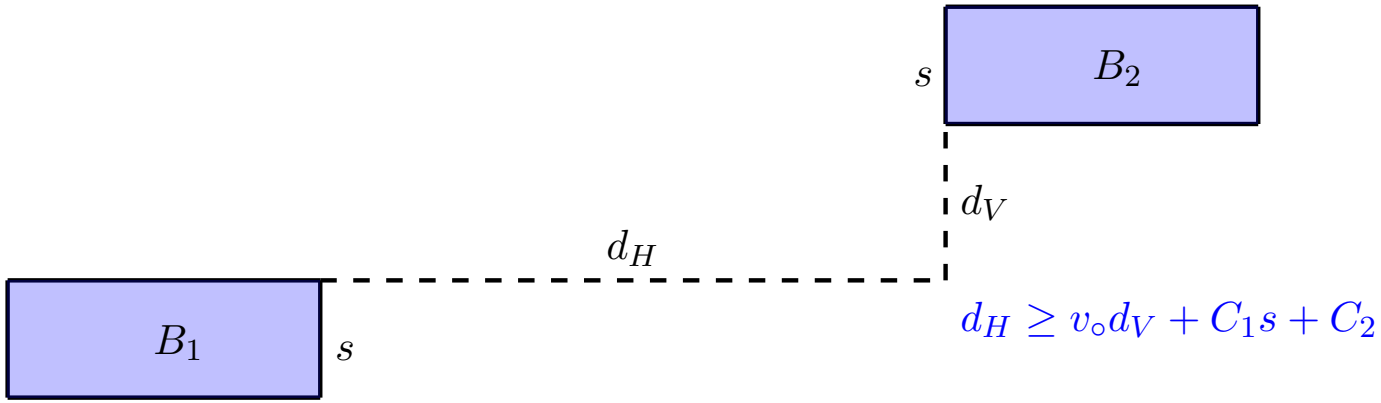
- Bound on rates: $\sup_{\xi \in E} \{\alpha(\xi) + \beta(\xi)\} \leq \lambda \in [1, \infty)$.

The model

Lateral Decoupling for the Environment

There exist positive constants $C_\circ$, $\kappa_\circ$, $v_\circ$, C_1, C_2 and $\gamma_\circ > 1$, s.t.

for every pair of boxes B_1, B_2:



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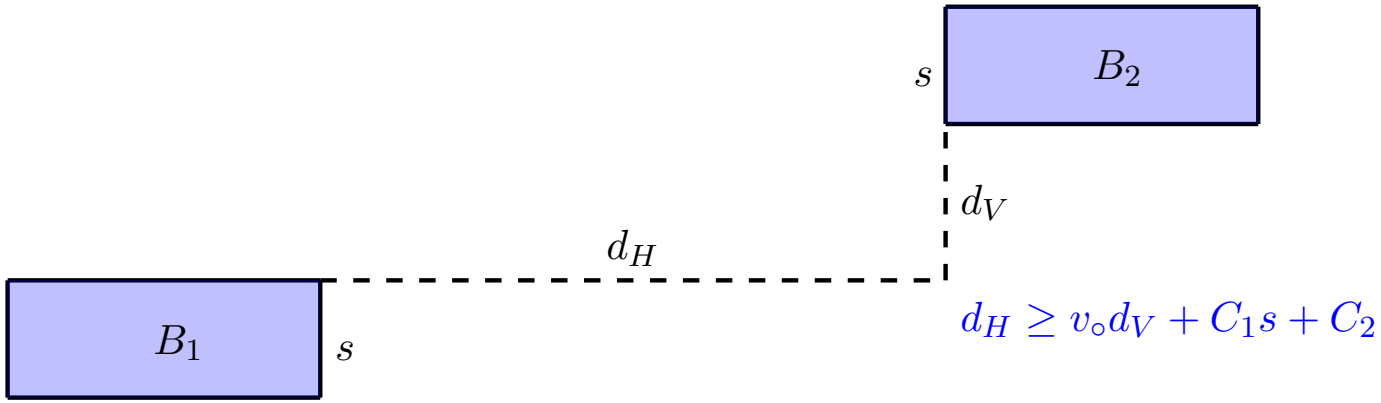

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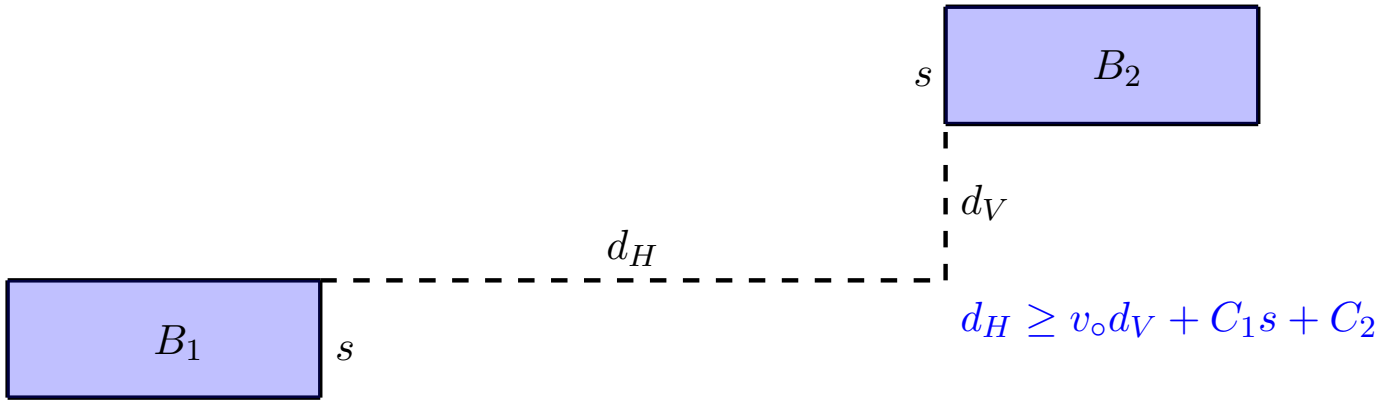

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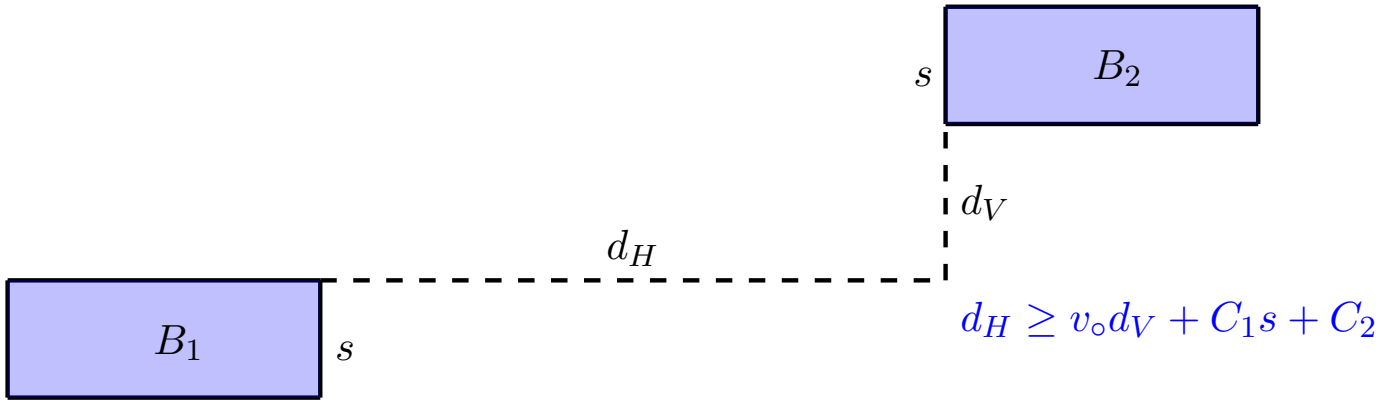

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Examples: zero-range, simple exclusion.

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- Non-nestling case: If

$$\inf_{\xi \in E} \{\alpha(\xi) - \beta(\xi)\} > v, \quad \text{for some } v > 0 \ ,$$

then ballisticity condition holds with $v_\star = v$.

Main theorem

Theorem (Arcanjo, Baldasso, H., dos Santos)

Assume lateral decoupling and ballisticity with $v_\star > v_\circ$, and let

$$\gamma = \min\{\gamma_\circ, \gamma_\star\}, \qquad \kappa = \min\{\kappa_\circ, \kappa_\star\}. \tag{1}$$

Then there exists a speed $v \geq v_\star$ and, for any $\varepsilon > 0$, a positive constant C_ε such that

$$\mathbb{P}\left(\exists\, t \geq T \colon |X_t - vt| \geq \varepsilon t\right) \leq C_\varepsilon e^{-c\kappa(\log^+ T)^\gamma} \qquad \text{for all } T \geq 0. \tag{2}$$

In particular,

$$\lim_{t\to\infty} \frac{X_t}{t} = v \quad \mathbb{P}\text{-almost surely.} \tag{3}$$

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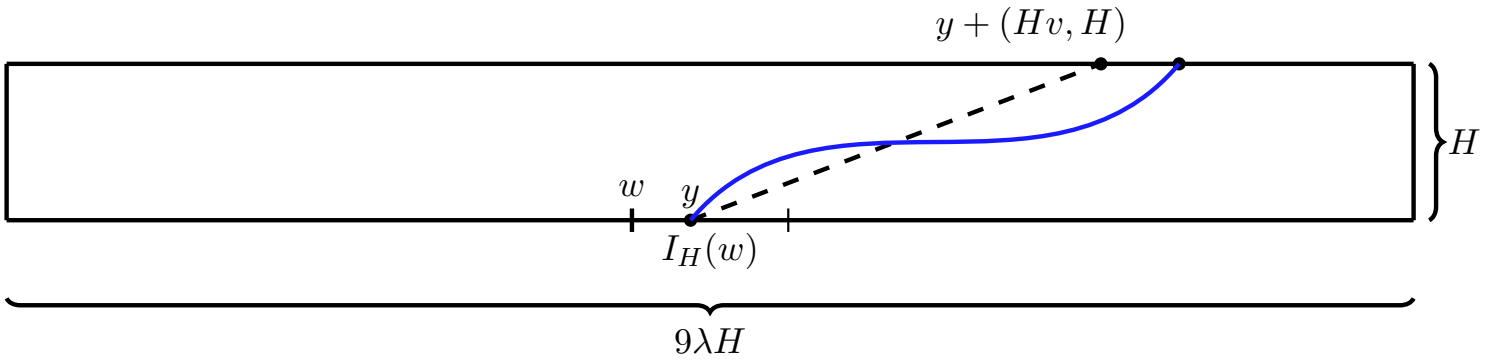

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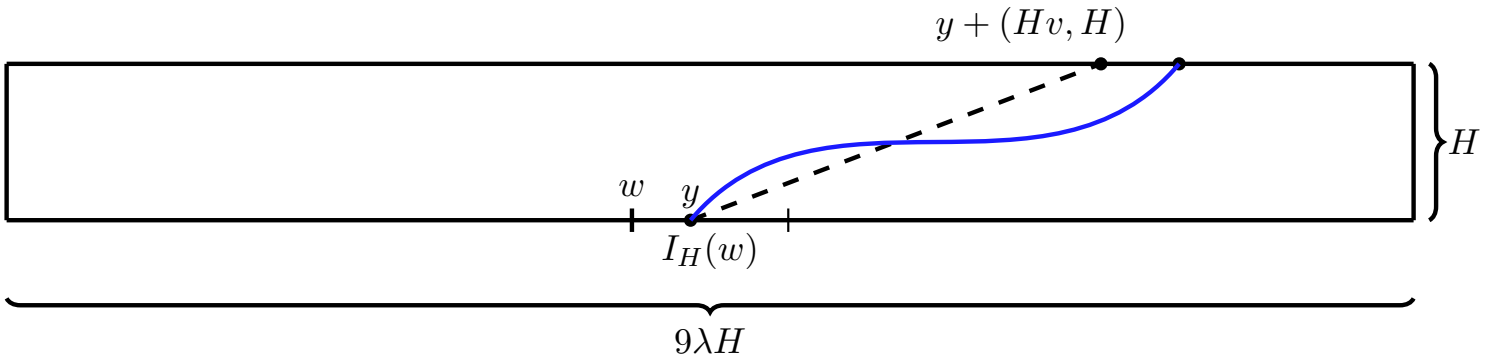

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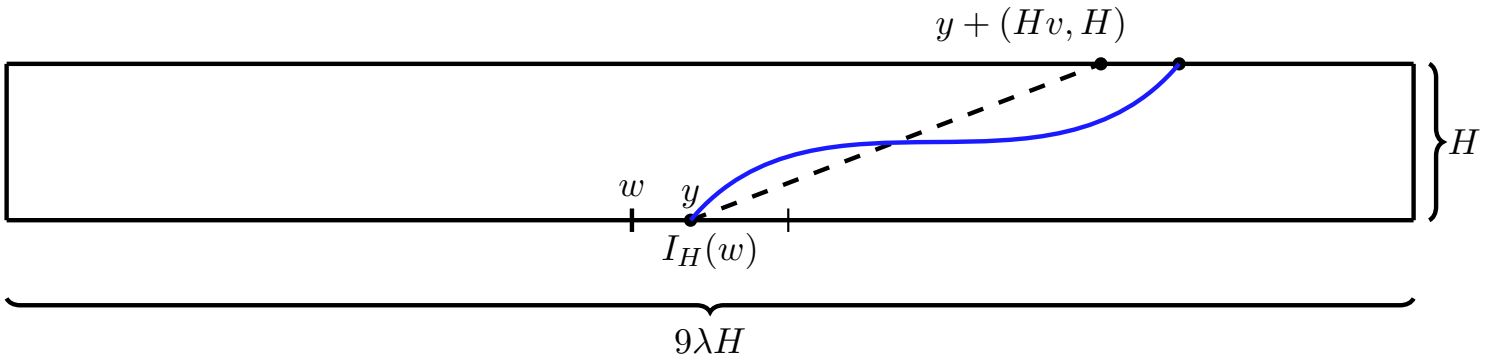

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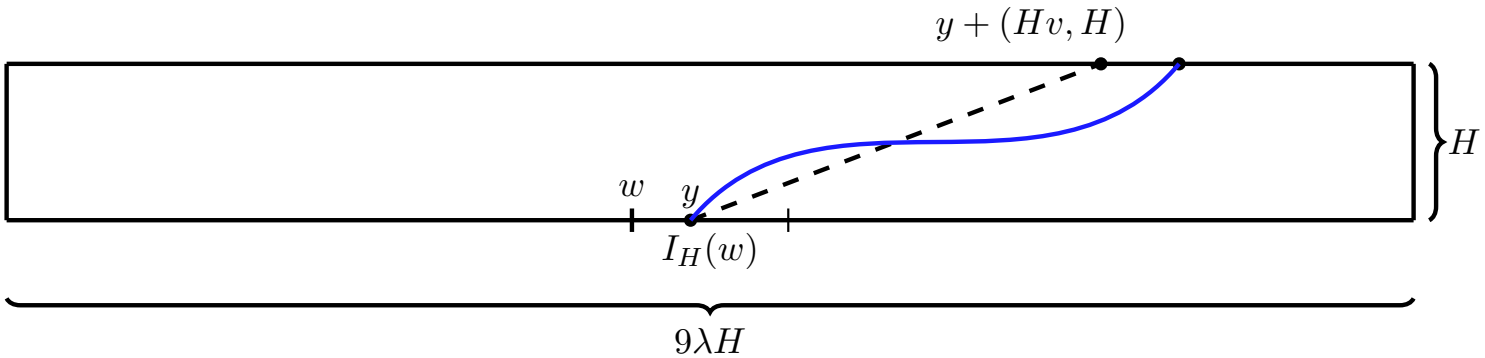

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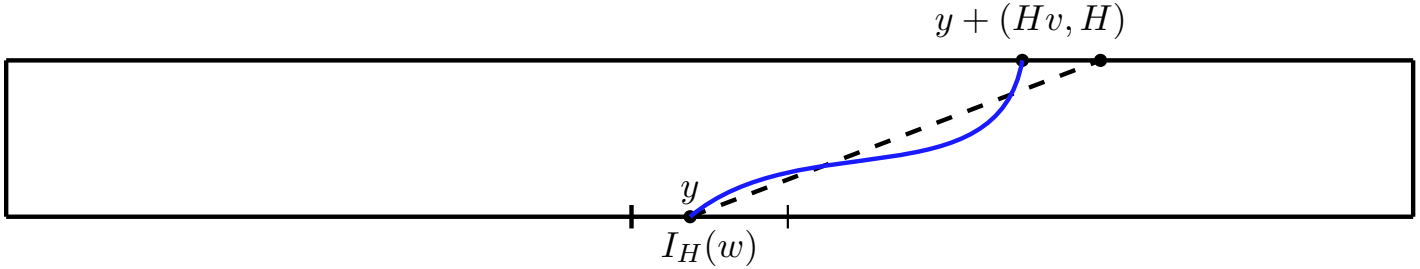

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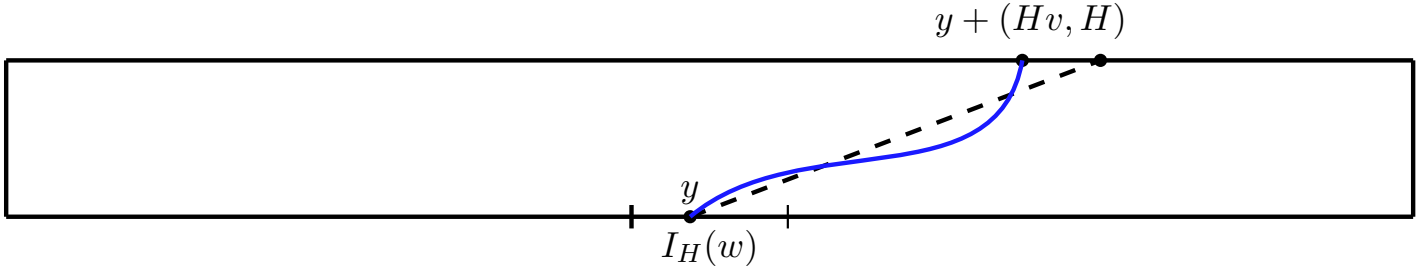

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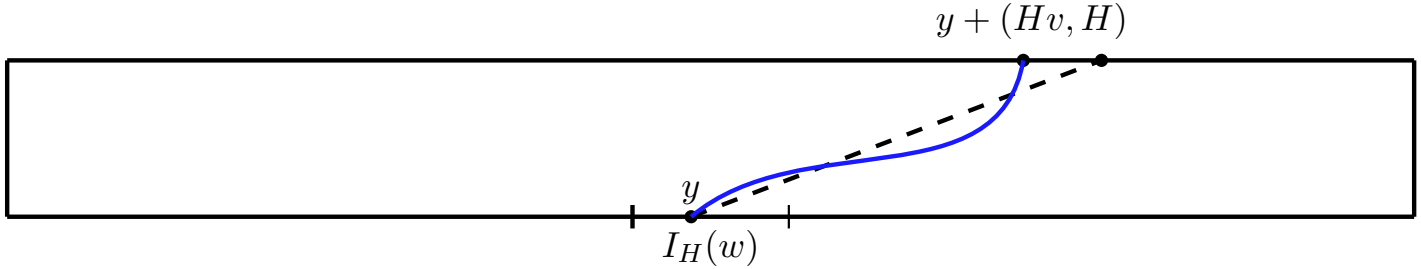

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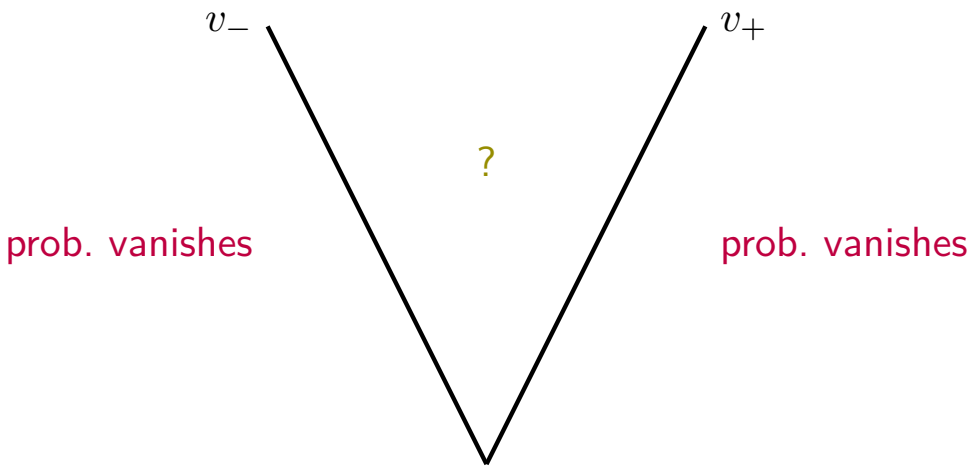

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Proposition

For every $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that

$$p_H(v_+ + \varepsilon) \leq ce^{-c'\kappa \log^\gamma H} \quad \textit{and} \quad \tilde{p}_H(v_- - \varepsilon) \leq ce^{-c'\kappa \log^\gamma H},$$

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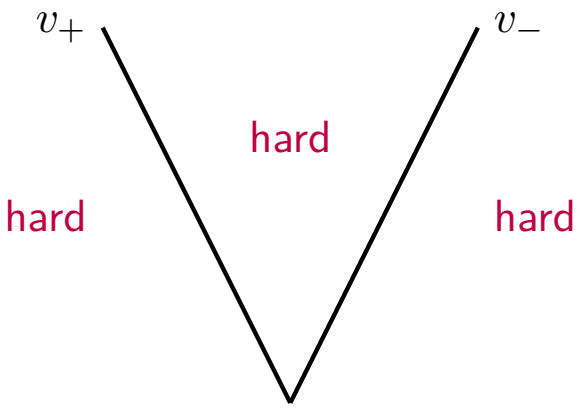

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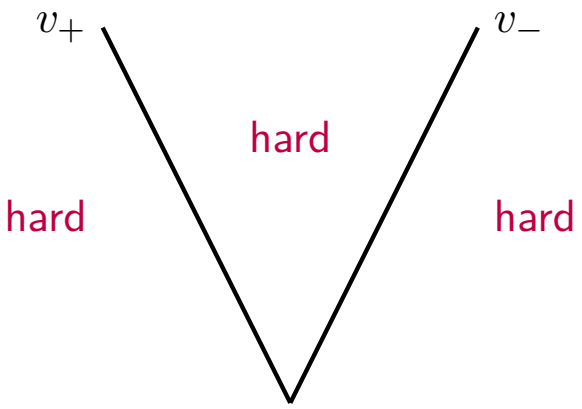
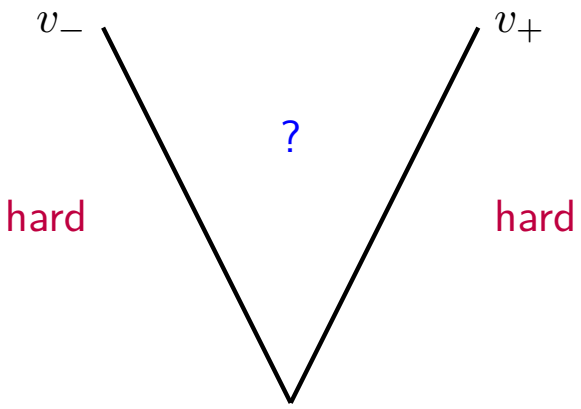




Fast decay above v_+ and below v_-

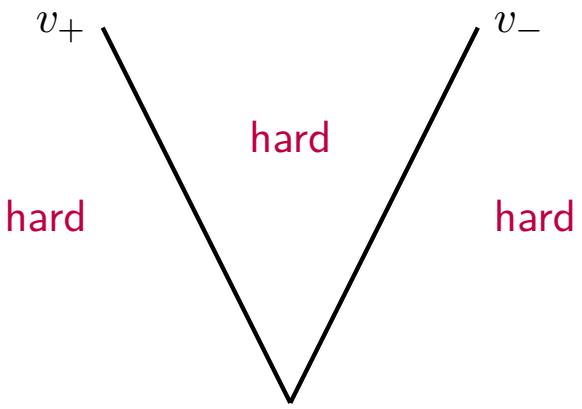
Bootstrap to get strong decay above v_+ and below v_-.

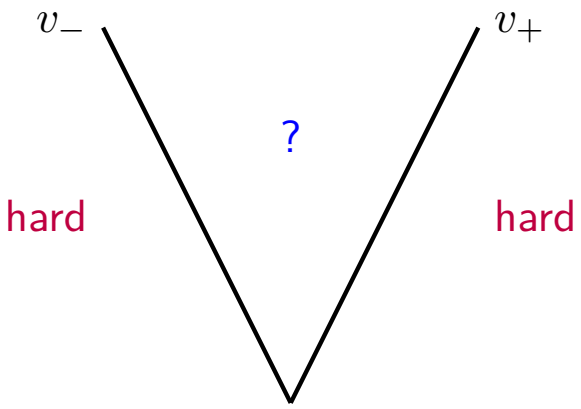
Proposition

For every $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that

$$p_H(v_+ + \varepsilon) \leq c e^{-c'\kappa \log^\gamma H} \quad \textit{and} \quad \tilde{p}_H(v_- - \varepsilon) \leq c e^{-c'\kappa \log^\gamma H},$$

for all $H \geq 1$.





Remark: implies $v_- \leq v_+$.

Coincidence of upper and lower speeds

Sharpness-type result: rule out the fictitious regime $v_- < v < v_+$.

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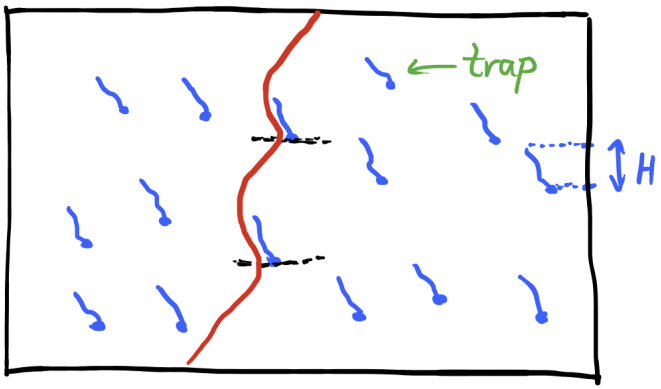

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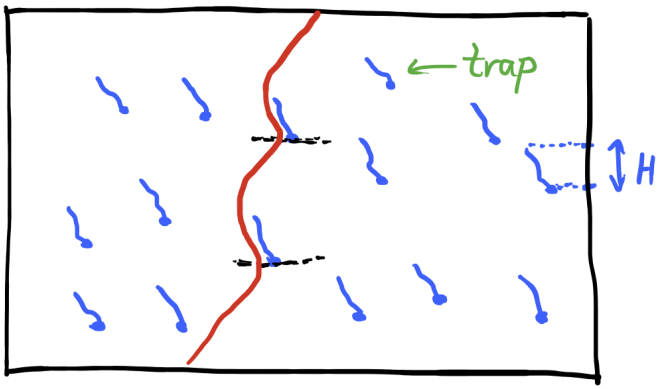

- Good proportion of time running with speed slightly above v_-.

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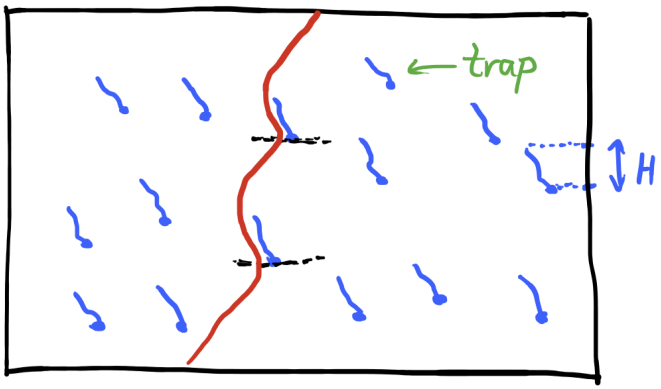

- Good proportion of time running with speed slightly above v_-.
- To catch up with v_+: needs to run faster then v_+ for another good proportion of its time.

Concluding remarks

- LLN under decoupling and ballisticity assumptions.
- Main technique: renormalization.
- No use of regeneration techniques.
- New for zero-range and asymmetric exclusion.
- Only works for $d = 1$, no CLT.

References

- ARCANJO, Weberson S., BALDASSO, R., HILÁRIO, Marcelo R., SANTOS, Renato. Law of large numbers for ballistic random walks in dynamic random environments under lateral decoupling. arXiv preprint arXiv:2205.00282.
- BALDASSO, Rangel; TEIXEIRA, Augusto. Spread of an infection on the zero range process. Ann. Inst. H. Poincaré Probab. Statist. 56(3): 1898-1928, 2020.
- BLONDEL, Oriane; HILÁRIO, Marcelo R.; TEIXEIRA, Augusto. Random walks on dynamical random environments with nonuniform mixing.The Annals of Probability, 48(4):2014 – 2051, 2020.
- HILÁRIO, Marcelo R.; KIOUS, Daniel; TEIXEIRA, Augusto. Random walk on the simple symmetric exclusion process. Communications in Mathematical Physics, v. 379, n. 1, p. 61-101, 2020.

Multiscale renormalization

Scales, boxes, deviation events

- Scales $(L_k)_{k\geq 0}$

$$L_0 := 10^{10} \quad \text{and} \quad L_{k+1} := L_k^{1+\nu}, \quad k \geq 0,\ \nu \in (0,1).$$

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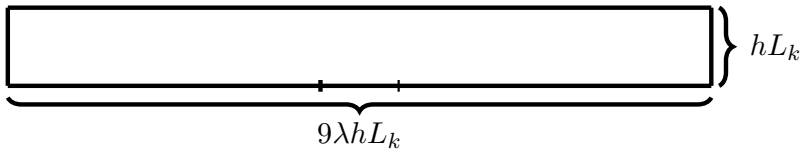

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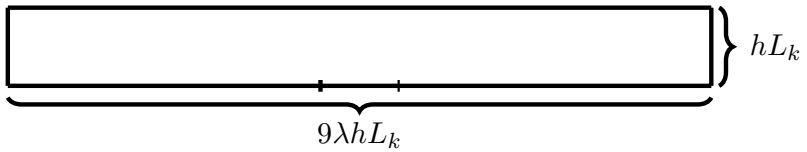


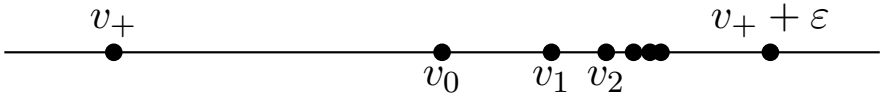
- Translates and deviation events

$$B^h_{L_k}(w) := w + B^h_{L_k} \longrightarrow A_{hL_k,w}(v) \quad \text{and} \quad \tilde{A}_{hL_k,w}(v).$$

Multiscale renormalization

Cascading property for deviation events

- $(v_k)_{k\geq 0}$ increasing:



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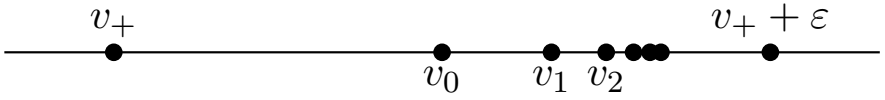

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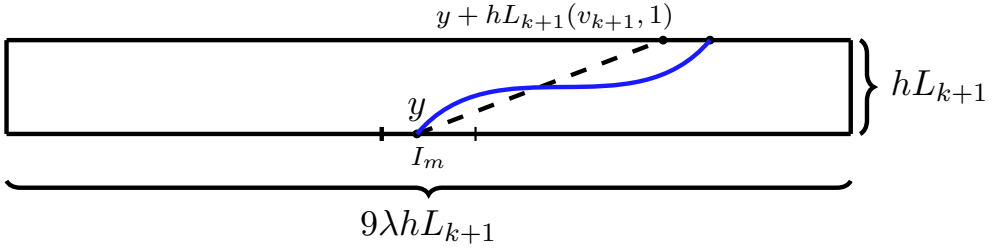

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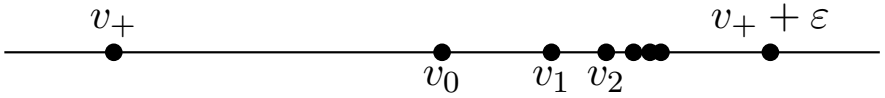

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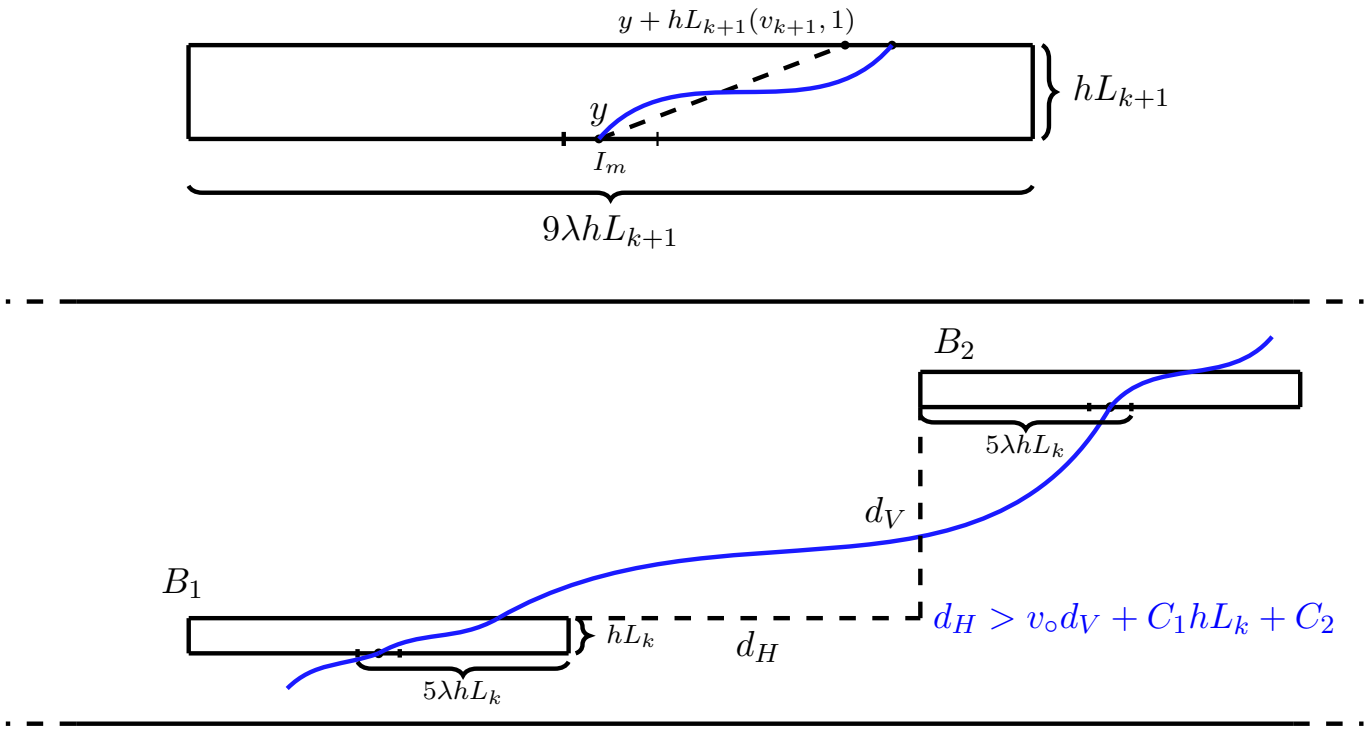

Multiscale renormalization

Strong decay above v_+

- **Cascading + Decoupling:**

$$p_{hL_{k+1}}(v_{k+1}) \leq c\, L_k^{4\nu}\, p^2_{hL_k}(v_k) + e^{-8\kappa(\log hL_k)^\gamma}$$

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- **Triggering:** Fix $h = h_\circ$ large enough

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