# Coproduct and amalgamation of deductive systems by means of ordered algebras

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### A brief overview

Starting point

• Propositional logic is the "deductive skeleton" of any logic.

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- (Concrete) language expansion.
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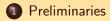
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- Joining together deductive systems with a common fragment.



### 2 Language expansions

3 Coproducts of deductive systems

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# Quantales and modules

#### Definition

A (unital) quantale  $(Q, \bigvee, \cdot, 1)$  is a complete residuated lattice. Quantale homomorphisms preserve arbitrary joins and the monoid structure.

A (left) Q-module  $(M, \bigvee)$  is a sup-lattice with a scalar multiplication  $\cdot : Q \times M \to M$  satisfying the usual module properties.

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#### Nuclei

A *Q*-module nucleus over *M* is a closure operator  $\gamma : M \to M$  such that  $a \cdot \gamma(x) \leq \gamma(a \cdot x)$  for all  $a \in Q$  and  $x \in M$ . Nuclei are in bijective correspondence with congruences. The image  $M_{\gamma}$  of  $\gamma$  is a *Q*-module with  $\gamma \bigvee X := \gamma(\stackrel{M}{\bigvee} X)$  and  $a \cdot_{\gamma} x := \gamma(a \cdot x)$ , for all  $a \in Q$ ,  $X \cup \{x\} \in M_{\gamma}$ .

### Propositional languages

#### Definition

A propositional language is a pair  $\mathcal{L} = (L, \nu)$ , where *L* is a set (of connectives) and  $\nu : L \to \omega$  is the arity function. The set  $Fm_{\mathcal{L}}$  of  $\mathcal{L}$ -formulas is defined in the usual manner from a denumerable set of variables  $Var = \{x_n \mid n < \omega\}$ ; it is the term algebra over  $\mathcal{L}$ . We denote by  $\Sigma_{\mathcal{L}}$  the monoid of substitutions of  $\mathcal{L}$ , i.e., of the endomorphisms of the  $\mathcal{L}$ -algebra  $Fm_{\mathcal{L}}$ .

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$$Seq_{T} = \{\varphi_{1}, \dots, \varphi_{m} \Rightarrow \psi_{1}, \dots, \psi_{n} \mid \varphi_{i}, \psi_{j} \in Fm_{\mathcal{L}}, (m, n) \in T\}$$

of  $\mathcal{L}$ -sequents closed under the types of T will be identified with  $\bigcup_{(m,n)\in T} Fm_{\mathcal{L}}^m \times Fm_{\mathcal{L}}^n$ .

### **Propositional logics**

#### Definition

By a propositional logic we mean a pair  $(D, \vdash)$ , where D – called the domain – is the set of formulas, the one of equations or a set of sequents closed under type over a propositional language  $\mathcal{L} = (L, \nu)$ , and  $\vdash$  is a binary relation on  $\mathcal{P}D$  satisfying, for all  $\Phi, \Psi, \Xi \in \mathcal{P}D$ , the following conditions:

- if  $\Psi \subseteq \Phi$ , then  $\Phi \vdash \Psi$ ;
- if  $\Phi \vdash \Psi$  and  $\Psi \vdash \Xi$ , then  $\Phi \vdash \Xi$ ;
- $\Phi \vdash \bigcup_{\Phi \vdash \Psi} \Psi;$
- $\Phi \vdash \Psi$  implies  $\sigma[\Phi] \vdash \sigma[\Psi]$  for each substitution  $\sigma \in \Sigma_{\mathcal{L}}$ .

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### (Not just) closure operators

#### The consequence operator

It is well-known that the mapping  $\gamma_{\vdash} : \Phi \in \mathcal{P}D \mapsto \bigcup_{\Phi \vdash \Psi} \Psi \in \mathcal{P}D$  is a closure operator on  $(\mathcal{P}D, \subseteq)$ , but this representation of deductive systems is somewhat unsatisfactory. Not every closure operator is induced by a consequence relation.

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#### Refining the representation [Galatos and Tsinakis 2009]

 $(\mathcal{P}D, \bigcup)$  is a left module over the quantale  $(\mathcal{P}\Sigma_{\mathcal{L}}, \bigcup, \cdot, \{id\})$  with the action  $(\Sigma, \Phi) \mapsto \Sigma \cdot \Phi = \{\sigma(\varphi) \mid \sigma \in \Sigma, \varphi \in \Phi\}$  and  $\gamma_{\vdash}$  is a  $\mathcal{P}\Sigma_{\mathcal{L}}$ -module nucleus on  $\mathcal{P}D$ , i.e. a closure operator such that  $\Sigma \cdot \gamma_{\vdash}(\Phi) \subseteq \gamma_{\vdash}(\Sigma \cdot \Phi)$ , whose image is the module of theories  $Th_{\vdash}$ of  $\vdash$ .

Moreover, the correspondence between nuclei and substitution invariant consequence relations on  $\mathcal{P}D$  is bijective.

Interpretations and translations

#### Theorem [Galatos and Tsinakis 2009]

A propositional deductive system  $(D, \vdash)$  over  $\mathcal{L}$  is interpretable in (resp.: representable in, equivalent to) another  $\mathcal{L}$ -system  $(D', \vdash')$  if and only if there exists a  $\mathfrak{P}\Sigma_{\mathcal{L}}$ -module homomorphism (resp.: embedding, isomorphism) from  $Th_{\vdash}$  to  $Th_{\vdash'}$ .

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#### Translations

In order to extend the previous result to the case of systems over two different languages  $\mathcal{L}$  and  $\mathcal{L}'$ , it is necessary to introduce the concept of language translation. A translation  $\tau : \mathcal{L} \to \mathcal{L}'$  turns out to induce a quantale morphism  $t : \mathcal{P}\Sigma_{\mathcal{L}} \to \mathcal{P}\Sigma_{\mathcal{L}'}$  (and we also have a complete characterization of quantale morphisms induced by a translation). [R. 2013]

### Restricting and extending the scalars

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A quantale morphism  $h: Q \to R$  turns every *R*-module into a *Q*-module. In fact it defines a functor  $()_h: R-\mathcal{M}od \to Q-\mathcal{M}od$  which has both a left and a right adjoint. An analogous procedure is known in the theory of ring modules as restricting the scalars along *h*.

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#### The left adjoint

The left adjoint to the functor  $()_h$  uses the tensor product:

 $()_h^l: M \in Q\text{-}\mathcal{M}od \mapsto R \otimes_Q M \in R\text{-}\mathcal{M}od.$ 

We shall call it the extension of scalars. Note:  $x \in M \mapsto 1 \otimes x \in R \otimes_Q M$  is a *Q*-module morphism.



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### Tensor product and language expansion

Let  $\mathcal{L}_1 \supseteq \mathcal{L}$  and  $i : \mathfrak{P}\Sigma_{\mathcal{L}} \to \mathfrak{P}\Sigma_{\mathcal{L}_1}$  be the quantale embedding associated to the inclusion map. Let  $(D, \vdash)$  be a d.s. over  $\mathcal{L}$ ,  $\gamma$  its associated nucleus, and *Th* its  $\mathfrak{P}\Sigma_{\mathcal{L}}$ -module of theories.

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 $\mathfrak{P}\Sigma_{\mathcal{L}_1} \otimes_{\mathfrak{P}\Sigma_{\mathcal{L}}} \mathfrak{P}Fm_{\mathcal{L}} \cong \mathfrak{P}Fm_{\mathcal{L}_1}, \text{ hence } \mathfrak{P}\Sigma_{\mathcal{L}_1} \otimes_{\mathfrak{P}\Sigma_{\mathcal{L}}} \mathfrak{P}D \text{ is isomorphic to the domain } \mathfrak{P}D_1 \text{ in the language } \mathcal{L}_1 \text{ of the same type of } \mathfrak{P}D.$ 

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#### Theorem

 There exists a consequence relation ⊢<sub>1</sub> on PD<sub>1</sub> whose PΣ<sub>L1</sub>-module of theories Th<sub>1</sub> is isomorphic to PΣ<sub>L1</sub>⊗<sub>PΣ<sub>L</sub></sub> Th.

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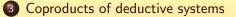
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- The mapping  $\Phi \in Th \mapsto \{id\} \otimes \Phi \in \mathfrak{P}\Sigma_{\mathcal{L}_1} \otimes_{\mathfrak{P}\Sigma_{\mathcal{L}}} Th$  is a  $\mathfrak{P}\Sigma_{\mathcal{L}}$ -module embedding.

### Preliminaries

### 2 Language expansions



#### 4 Amalgamation of logics

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### Notations

For i = 1, 2, let  $\mathcal{L}_i$  be two languages,  $(D_i, \vdash_i)$  be  $\mathcal{L}_i$ -deductive systems of the same type, and  $\mathcal{L}$  be the disjoint union of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . Then  $\mathcal{P}\Sigma_{\mathcal{L}} \cong \mathcal{P}\Sigma_{\mathcal{L}_1} \amalg \mathcal{P}\Sigma_{\mathcal{L}_2}$ . Let also E be the  $\Sigma_{\mathcal{L}}$ -domain of the same type of the two given systems, and let us set the following:

- $\gamma_i$  nuclei associated to  $\vdash_i$ ,
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- $\gamma'_i: \Phi \in \mathfrak{P}E \mapsto \gamma_i(\Phi \cap D_i) \cup (\Phi \setminus D_i) \in \mathfrak{P}E$ , and

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- $\gamma'_i$ :  $\Phi \in \mathfrak{P}E \mapsto \gamma_i(\Phi \cap D_i) \cup (\Phi \setminus D_i) \in \mathfrak{P}E$ , and
- $\delta_i$  be the nuclei associated to the consequence relations  $\vdash_{\delta_i}$  on *E* defined by means of the axioms and rules of  $\vdash_i$ .

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### Notations

For i = 1, 2, let  $\mathcal{L}_i$  be two languages,  $(D_i, \vdash_i)$  be  $\mathcal{L}_i$ -deductive systems of the same type, and  $\mathcal{L}$  be the disjoint union of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . Then  $\mathcal{P}\Sigma_{\mathcal{L}} \cong \mathcal{P}\Sigma_{\mathcal{L}_1} \amalg \mathcal{P}\Sigma_{\mathcal{L}_2}$ . Let also E be the  $\Sigma_{\mathcal{L}}$ -domain of the same type of the two given systems, and let us set the following:

- $\gamma_i$  nuclei associated to  $\vdash_i$ ,
- $Th_i = (\mathcal{P}D_i)_{\gamma_i}$  the corresponding modules of theories,
- $d_i: \mathcal{P}D_i \to \mathcal{P}E$  the inclusion maps,
- $\gamma'_i$ :  $\Phi \in \mathfrak{P}E \mapsto \gamma_i(\Phi \cap D_i) \cup (\Phi \setminus D_i) \in \mathfrak{P}E$ , and
- $\delta_i$  be the nuclei associated to the consequence relations  $\vdash_{\delta_i}$  on *E* defined by means of the axioms and rules of  $\vdash_i$ .

Last, let 
$$\delta = \delta_1 \vee \delta_2$$
, and  $Th = (\mathcal{P}E)_{\delta}$ .

### The embedding party

#### Lemma 1

Let  $i \neq k \in \{1,2\}$ . Then, for all  $\Phi \cup \{\psi\} \in \mathcal{P}D_k$ ,  $\psi \in \delta_i(\Phi)$  if and only if  $\psi \in \Phi$ .

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#### Proposition 2

There exist  $\mathcal{P}\Sigma_{\mathcal{L}i}$ -module embeddings of  $Th_i$  and  $\mathcal{P}\Sigma_{\mathcal{L}k}$ -module embeddings of  $\mathcal{P}D_k$  into  $Th_{\delta_i}$ ,  $i \neq k \in \{1, 2\}$ :

 $f_i: \Phi \in Th_i \mapsto \delta_i(\Phi) \in Th_{\delta_i}, \text{ and } g_k: \Phi \in \mathcal{P}D_k \mapsto \delta_i(\Phi) \in Th_{\delta_i}.$ 

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#### Theorem 3

There exist  $\mathfrak{P}\Sigma_{\mathcal{L}i}$ -module embeddings  $e_i$ :  $Th_i \to Th$ , i = 1, 2.

### The embedding party

#### Theorem 4

For i = 1, 2, the  $\mathcal{P}\Sigma_{\mathcal{L}}$ -modules  $\mathcal{P}\Sigma_{\mathcal{L}} \otimes_{\mathcal{P}\Sigma_{\mathcal{L}}i} Th_i$  and  $Th_{\delta_i}$  are isomorphic.

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#### Theorem 5

The coproduct of  $Th_1$  and  $Th_2$  embeds as a sup-lattice in Th.

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## The embedding party

#### Theorem 4

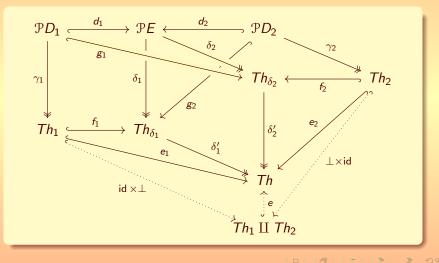
For i = 1, 2, the  $\mathcal{P}\Sigma_{\mathcal{L}}$ -modules  $\mathcal{P}\Sigma_{\mathcal{L}} \otimes_{\mathcal{P}\Sigma_{\mathcal{L}}i} Th_i$  and  $Th_{\delta_i}$  are isomorphic.

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- Each (D<sub>i</sub>,⊢<sub>i</sub>) is representable in (E,⊢<sub>δ</sub>) (Theorem 3 and [R. 2013, Theorem 7.1]).
- *Th*<sub>δi</sub> is precisely the result of a language expansion (via tensor product) on *Th<sub>i</sub>* (Theorem 4).
- The sup-lattice of theories of δ contains an isomorphic copy of the coproduct of the *Th<sub>i</sub>*'s (Theorem 5).

## The embedding fireworks





### 4 Amalgamation of logics

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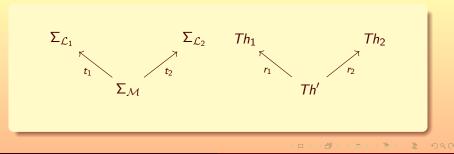
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## The V-formations

With the same notations, let us add another language  $\mathcal{M}$  and a deductive system  $(\mathcal{C}, \vdash_{\beta})$  on  $\mathcal{M}$  (again, of the same type of the  $D_i$ 's), with associated nucleus  $\beta$  and module of theories  $Th' = (\mathcal{P}\mathcal{C})_{\beta}$ . Let us also assume the existence of translations  $\tau_i : \mathcal{M} \to \mathcal{L}_i$  and structural representations  $r_i : Th' \to Th_i$  via  $\tau_i$ .

## The V-formations

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## Translations and more nuclei

Thanks to the translation morphisms  $t_i$  and the inclusion ones of  $\mathcal{P}\Sigma_{\mathcal{L}i}$  into  $\mathcal{P}\Sigma_{\mathcal{L}}$ , all of the modules and embeddings which appeared in the previous section, including e, are now in  $\mathcal{P}\Sigma_{\mathcal{M}}$ - $\mathcal{M}od$ .

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Let  $\varepsilon$  be the  $\mathcal{P}\Sigma_{\mathcal{L}}$ -module nucleus on  $\mathcal{P}E$  determined by the union of axioms and rules of  $\vdash_1$  and  $\vdash_2$ , and the set of rules

$$\Theta = \left\{ \frac{e_i r_i(\{\varphi\})}{e_k r_k(\{\varphi\})} \middle| \varphi \in C, i \neq k \in \{1, 2\} \right\}.$$

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$$\Theta = \left\{ \frac{e_i r_i(\{\varphi\})}{e_k r_k(\{\varphi\})} \middle| \varphi \in C, i \neq k \in \{1, 2\} \right\}.$$

Let  $\zeta$  be the nucleus determined by  $\Theta$ , and  $\varepsilon_i = \delta_i \lor \zeta$ . We have:

$$\varepsilon = \delta \lor \zeta = \delta_1 \lor \delta_2 \lor \zeta = \varepsilon_1 \lor \varepsilon_2.$$

## The amalgamation party

### Lemma 6

There exist  $\mathcal{P}\Sigma_{\mathcal{L}i}$ -module embeddings of  $Th_i$  into  $Th_{\varepsilon_i}$ .

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There exist  $\mathfrak{P}\Sigma_{\mathcal{L}i}$ -module embeddings  $m_i: Th_i \to Th_{\varepsilon}$ .

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#### Theorem 8

There exist  $\mathfrak{P}\Sigma_{\mathcal{L}}$ -module nuclei  $\zeta_i$  on  $\mathfrak{P}\Sigma_{\mathcal{L}} \otimes_{\mathfrak{P}\Sigma_{\mathcal{L}}i} Th_i$  such that the  $\mathfrak{P}\Sigma_{\mathcal{L}}$ -modules  $(\mathfrak{P}\Sigma_{\mathcal{L}} \otimes_{\mathfrak{P}\Sigma_{\mathcal{L}}i} Th_i)_{\zeta_i}$  and  $Th_{\varepsilon_i}$  are isomorphic.

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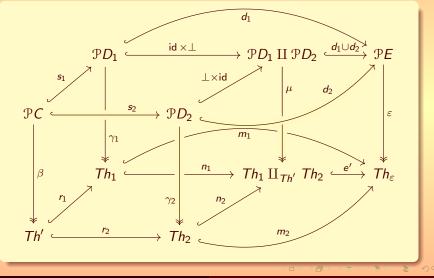
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#### Theorem 9

The amalgamated coproduct of the  $\mathcal{P}\Sigma_{\mathcal{M}}$ -modules  $Th_1$  and  $Th_2$ w.r.t. Th' embeds in  $Th_{\varepsilon}$ .

## The amalgamation fireworks



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# Thank you!

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