

# Large Deviations for clustering observations

Let  $(Y_i)_{i \in \mathbb{N}}$ , stationary, ergodic

Let  $A \in \sigma(Y_0^{k-1})$

Let  $X_i = \mathbb{1}_{T^{-i}(A)}$ , where  $T$  shift.  $p = \mathbb{P}(A)$ .

- ▶ Ergodic Theorem

$$\frac{\sum_{i=1}^n X_i}{n} \rightarrow p \quad \text{in prob, } \mathbb{P} - q.c.$$

- ▶ Central Limit Th.

$$\frac{\sum_{i=1}^n X_i/n - p}{p(1-p)/\sqrt{n}} \Rightarrow \mathcal{N}(0, 1)$$

- ▶ Concentration inequalities                              Large Deviations

$$\mathbb{P}\left(\left|\frac{\sum_{i=1}^n X_i}{n} - p\right| > \epsilon\right) \leq e^{-f(\epsilon)n} \quad \lim_{n \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}\left(\left|\frac{\sum_{i=1}^n X_i}{n} - p\right| > \epsilon\right) = I(\epsilon)$$

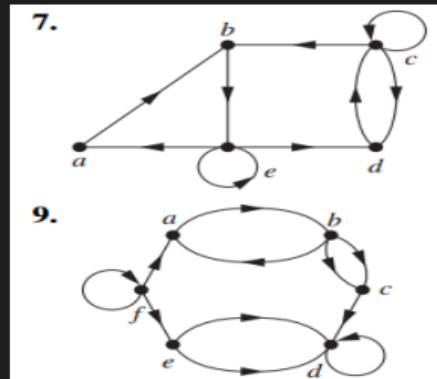
## Previous results for L.D.

- ▶ independent
  - ▶ Bernstein (Radamacher)  $\exp(-\epsilon^2 n / 2[1 + (1/3)\epsilon])$   
(bounded by  $U$ )  $\exp(-\epsilon^2 n / 2[M_2 + (U/3)\epsilon])$
  - ▶ Chernoff ( $X \in [0, 1]$ )  $\exp(-\epsilon^2 n / 2)$
  - ▶ Hoeffding (bounded)  $\exp(-2\epsilon^2 n / U^2)$
- ▶ dependent
  - ▶ Martingale
  - ▶ negative correlated

# Grafo

Clustering observable

Example



## Clustering observable

Example  $A = (0, 1, 0, 0, 1)$

0 1 0 0 1

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|   |   |   |   |   |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 |

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## Chernoff Method

$$\mathbb{P}\left(\sum_{i=1}^n X_i > (p + \epsilon)n\right) \leq e^{-(p+\epsilon)\lambda n} \mathbb{E}(e^{\lambda N_0^{n-1}})$$

### Problems in the dependent case

- ▶ Control  $\mathbb{E}(e^{\lambda N_0^{n-1}})$
- ▶ Find (approx)  $\lambda_{\min}$
- ▶ Compute the bound

## Theorem

Theorem Process:  $\psi$ -mixing. For all  $q \in \mathbb{N}$

$$\mathbb{P}\left(\frac{\sum_{i=1}^n X_i}{n} > p + \epsilon\right) \leq \exp\left(-\frac{\epsilon^2 n}{pK}\right) + (q-1) \exp\left(-\epsilon^2 N K_{A,\epsilon}\right)$$

## Structure of the proof

- ▶ Spectral analysis of recursive equation
- ▶ Use the independent case information
- ▶ Quadratic interpolation (not linear)

## Sketch of proof

Condition on  $X_0$

$$\begin{aligned}\mathbb{E}(e^{\lambda N_n}) &= pe^\lambda \mathbb{E}(e^{\lambda N_1^{n-1}} | I_0 = 1) + (1-p)\mathbb{E}(e^{\lambda N_1^{n-1}} | I_0 = 0) \\ &= \mathbb{E}(e^{\lambda N_1^{n-1}}) + p(e^\lambda - 1)\mathbb{E}(e^{\lambda N_1^{n-1}} | I_0 = 1).\end{aligned}$$

Define  $\tau$  as the tail of the cluster

$$N_1^{n-1} = N_1^\tau + N_{\tau+q}^{n-1}.$$

Thus

$$\mathbb{E}(e^{\lambda N_1^{n-1}} | I_0 = 1) \leq \mathbb{E}(e^{\lambda N_1^\tau} | I_0 = 1) (1 + \psi(q)) \mathbb{E}(e^{\lambda N_{\tau+q}}).$$

Write  $x_n \leq x_{n-1} + x_{n-q} \underbrace{(1 + \psi(q))p(e^\lambda - 1)M(\lambda)}_{F(\lambda)}$

$$x_n = \sum_{i=1}^q C_i r_i^t$$

$r_i$ 's raízes complexos do polinômio associado

Control:  $r_i$ 's Rouché Theorem

$$r_1 \approx 1 + F(\lambda)$$

For  $i \geq 2$ , one has  $|r_i| \approx F(\lambda)^{1/q}$

Example

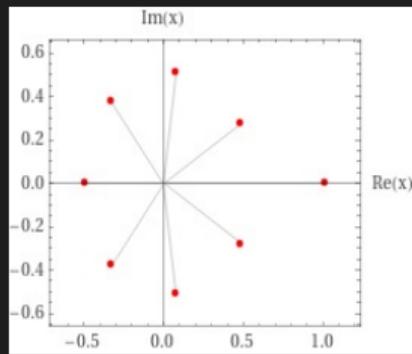


Figure :  $x^8 - x^7 - 0.01$

**Lemma:** For

$$\lambda \in [0, \lambda_0] = \ln \frac{p + \epsilon}{p} \frac{1 - p}{1 - p - \epsilon} \leq \frac{\epsilon(1 - \epsilon)}{p(1 - p - \epsilon)}$$

$$M(\lambda) \leq 1 + K_1 \lambda \leq 1 + K_2 \lambda \leq 1 + K_3 \lambda ,$$

**with**

- ▶  $K_1 = \mathbb{E}\left(\frac{e^{\lambda_0 N_1^\tau} - 1}{\lambda_0} | I_0 = 1\right)$
- ▶  $K_2 = \mathbb{E}(N_1^\tau e^{\lambda_0 N_1^\tau} | I_0 = 1)$
- ▶  $K_3 = \|\mathbb{E}(N_1^\tau | I_0 = 1)\|_r \|M(\lambda_0) | I_0 = 1\|_s \quad \text{for any } r, s \text{ conjugates.}$

Control:

$C_i$  resolução de  $VC = I$ , onde

$$C = (C_1, \dots, C_n)$$

$I = (x_0, \dots, x_{n-1})$  vetor de condições iniciais

$V$  matriz de Vandermonde( $r_i$ ),

$$V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_n \\ r_1^2 & r_2^2 & \dots & r_n^2 \\ .. & .. & \dots & .. \\ r_1^{n-1} & r_2^{n-1} & \dots & r_n^{n-1} \end{bmatrix}$$

## Grandes desvios

Theorem [Csiszár (2002)] Process  $\psi$ -mixing. For  $\epsilon > p\psi(q)$

$$\mathbb{P}\left(\frac{\sum_{i=1}^n X_i}{n} > p + \epsilon\right) \leq q \exp\left(-\frac{\epsilon^2 n}{p q}\right)$$

Theorem [Abadi (2023)] Process  $\psi$ -mixing

$$\mathbb{P}\left(\frac{\sum_{i=1}^n X_i}{n} > p + \epsilon\right) \leq \exp\left(-\frac{\epsilon^2 n}{pK}\right) + (q - 1) \exp\left(-\epsilon^2 n K_{A,\epsilon}\right)$$

## Generalização do Lemma de Kac

Lema [Kac, B. Am. Math. S.(1947)] Processo estacionário e ergódico,  $\mathbb{P}(A) > 0$ . Então

$$\mathbb{E}(\tau_A | A) = \frac{1}{\mathbb{P}(A)}$$

Tamanho médio do Cluster (C) **dado** que o Cluster começou

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Para todo  $q \geq T(A)$

$$\mathbb{E}(C_e | A \cap S^e(A^c)) = \frac{1}{\theta_e}$$

