VIII Encontro da Pós-Graduação em Matemática da UFBA
20 a 24 de Novembro de 2023

Coproduct and amalgamation of deductive systems by means of ordered algebras*

Ciro Russo †
UFBA

Abstract

An approach to abstract deductive systems by means of quantale modules was proposed by N. Galatos and C. Tsinakis in [3] and eventually developed by the present author [5, 6, 8]. However, apart from occasional references to logic appeared, for example, in [1, 2, 4], many concrete applications of the order-theoretic apparatus are still lacking.

More precisely, in [3] the authors showed that propositional deductive systems can be fully described by means of their lattices of theories which are modules over the powerset quantales of the monoids of substitutions of their languages. Among other things, they proved that, given two deductive systems over the same language, the existence of a quantale module homomorphism between the corresponding modules of theories is a sufficient condition for the existence of an interpretation of one system into the other. The present author extended that result in two directions [6], namely, to the case of systems with different underlying languages and to both weaker and stronger forms of interpretations.

Further possibile applications were pointed out by all of us in those papers, but most of them have not been concretely explored in details. In the present talk, we shall precisely address this issue by proving the results hereafter briefly summarized.

Concerning the algebraic tools, we shall only extend to quantale modules the approach to congruences by means of the so-called saturated elements, already known for quantale congruences.

Definition 1. Let Q be a quantale, M a Q-module, and ϑ be a binary relation on M. An element s of M is called *ϑ-saturated* if, for all $(v, w) \in \vartheta$ and $a \in Q$, $av \leq s \iff aw \leq s$. Let M_ϑ be the set of ϑ-saturated elements of M.

Theorem 2. *Let M be a Q-module, $\vartheta \subseteq M^2$, and $\rho_\vartheta : v \in M \mapsto \bigwedge\{s \in M_\vartheta \mid v \leq s\} \in M$.*

*This talk is based on the paper [9], with the same title, for which the author was awarded the 2021 "Newton da Costa Prize for Logic", from the Brazilian Logic Society, in January 2022, and won the "2nd World Logic Prizes Contest", at the 7th World Congress on Universal Logic, in Crete (Greece), in April 2022.

†e-mail: `ciro.russo@ufba.br`

Then ρ_ϑ is a Q-module nucleus whose image is M_ϑ. Moreover, M_ϑ, with the structure induced by ρ_ϑ, is isomorphic to the quotient of M w.r.t. the congruence generated by ϑ.

According to [6, Theorem 6.7], each quantale morphism $h : Q \to R$ induces an adjoint and co-adjoint functor $(\ )_h : R\text{-}\mathcal{M}od \to Q\text{-}\mathcal{M}od$, whose left adjoint is $R \otimes_Q \_$. Such a result suggested the possibility of expanding the language of a deductive system, but the main problem was to maintain the possibility of recovering the original lattice of theories within the new one. The expansion is an obvious application of the tensor products, while the preservation of the original system is the content of the following result.

Theorem 3. *Let $\mathcal{L}_1$ be an expansion of $\mathcal{L}$, $i : \mathscr{P}(\Sigma_\mathcal{L}) \to \mathscr{P}(\Sigma_{\mathcal{L}_1})$ the natural embedding, and $(D, \vdash)$ a deductive system in the language $\mathcal{L}$ whose module of theories shall be denoted by Th. Then Th is isomorphic to a $\mathscr{P}(\Sigma_\mathcal{L})$-submodule of $(\mathscr{P}(\Sigma_{\mathcal{L}_1}) \otimes_{\mathscr{P}(\Sigma_\mathcal{L})} Th)_i$.*

The failure of the amalgamation property for quantales is an obvious consequence of the one for semigroups and monoids. However, we prove that quantales of substitutions of propositional languages can actually be amalgamated, even in a very easy way.

Proposition 4. *Let $\mathcal{L}_1 = (L_1, \nu_1)$ and $\mathcal{L}_2 = (L_2, \nu_2)$ be propositional languages with a common fragment $\mathcal{L} = (L_1 \cap L_2, \nu)$, where $\nu = \nu_1\restriction_{L_1 \cap L_2} = \nu_2\restriction_{L_1 \cap L_2}$. Hence we have an amalgam of monoids $\Sigma_{\mathcal{L}_1} \xleftarrow{i_1} \Sigma_\mathcal{L} \xrightarrow{i_2} \Sigma_{\mathcal{L}_2}$ which is strongly embeddable in $\Sigma_{\mathcal{L}_1 \cup \mathcal{L}_2}$.*

Consequently, the quantale $\mathscr{P}(\Sigma_{\mathcal{L}_1 \cup \mathcal{L}_2})$ is the (strong) amalgamated coproduct of $\mathscr{P}(\Sigma_{\mathcal{L}_1})$ and $\mathscr{P}(\Sigma_{\mathcal{L}_2})$ w.r.t. $\mathscr{P}(\Sigma_\mathcal{L})$.

Last, we will show various techniques for combining different deductive systems thus obtaining a new one which encompasses the consequence relations of all of the initial systems. In particular, we will prove that the whole machinery is flexible enough not only to handle different situations but also to propose alternative constructions for each single case. We can briefly resume such constructions as follows

- Two systems with the same underlying language can always be merged either using $\mathcal{Q}$-module (possibly amalgamated) coproduct or by "doubling" the language. Module coproduct gives a system on some kind of two-sorted syntactic constructs and has the advantage of working in a pretty standard way, in the sense that its concrete implementation does not depend on the particular systems at hand. On the other hand, the construction based on the coproduct of quantales is definitely more effective when dealing with systems of the same type (both on formulas, equations, or sequents).
- Merging systems over different languages is obviously more laborious using module (possibly amalgamated) coproduct. On the contrary, the construction with quantale coproducts is essentially the same as the single-language case, but the capacity of taking into account possible isomorphic pieces of the modules of theories depends on the existence or not of some (possibly partial) translation and interpretation.

Besides the obvious interest for algebraic logicians, a standard method for merging different deductive systems may be useful in various applications in automated reasoning, such as automated theorem provers or decision-making processes.

References

[1] Cintula, P., Gil-Férez, J., Moraschini, T., and Paoli, F.; An abstract approach to consequence relations. *The Review of Symbolic Logic*, **12**(2) (2019), 331–371.

[2] Font, J. M.; *Abstract Algebraic Logic – An Introductory Textbook*. College Publications, ISBN 978-1-84890-207-7, 2006.

[3] Galatos, N., and Tsinakis, C.; Equivalence of consequence relations: an order-theoretic and categorical perspective. *Journal of Symbolic Logic* **74**/3 (2009), 780–810.

[4] Moraschini, T.; The semantic isomorphism theorem in abstract algebraic logic. *Annals of Pure and Applied Logic* **167**/12 (2016), 1298–1331.

[5] Russo, C.; Quantale Modules and their Operators, with Applications. *Journal of Logic and Computation* **20**/4 (2010), 917–946.

[6] Russo, C.; An order-theoretic analysis of interpretations among propositional deductive systems. *Annals of Pure and Applied Logic* **164** (2) (2013), 112–130.

[7] Russo, C.; Corrigendum to "An order-theoretic analysis of interpretations among propositional deductive systems" [Ann. Pure Appl. Logic 164 (2) (2013) 112–130]. *Annals of Pure and Applied Logic* **167** (3) (2016), 392–394.

[8] Russo, C.; Quantales and their modules: projective objects, ideals, and congruences. *South American Journal of Logic* **2**/2 (2016), 405–424.

[9] Russo, C.; Coproduct and amalgamation of deductive systems by means of ordered algebras. *Logica Universalis* **16** (2022), 355–380.

Tipo de Apresentação: Palestra