Intrinsic ergodicity for a certain class of Derived from Anosov

Cristina Lizana Araneda (UFBA)
Joint w/ L. Parra (PUCV) and C. Vásquez (PUCV)

VIII Encontro da Pós-graduação em Matemática da UFBA

Salvador, 21 de novembro de 2023

Setting

C^1-partially hyperbolic diffeomorphisms on compact boundaryless manifolds

Goal

To study the topological entropy inside the class of partially hyperbolic diffeomorphisms isotopics to Anosov and the existence (and uniqueness) of measure of maximal entropy.

Preliminaries

Definition (Dominated Splitting)

For an f-invariant set Λ a Df-invariant splitting of the tangent bundle

$$T_\Lambda M = E_1 \oplus \cdots \oplus E_k$$

is *dominated* if each bundle has constant dimension (at least two of them non-zero) and there exists an integer $\ell \geq 1$ such that for every $x \in \Lambda$, all $i = 1, \ldots, k-1$, and every pair of unitary vectors $u \in E_1(x) \oplus \cdots \oplus E_i(x)$ and $v \in E_{i+1}(x) \oplus \cdots \oplus E_k(x)$,

$$\frac{Df_x^\ell(u)}{Df_x^\ell(v)} \leq \frac{1}{2}.$$

Definition (Partially Hyperbolic)

$f \in Diff(M)$ is *partially hyperbolic* if there exists a dominated splitting

$$TM = E^s \oplus E^c \oplus E^u,$$

where E^s is uniformly contracting, E^u is uniformly expanding, and at least one of E^s and E^u is nontrivial.

Definition (Hyperbolic)

A partially hyperbolic diffeomorphism is *hyperbolic (or Anosov)* if

$$TM = E^s \oplus E^u.$$

(X, dist) metric space and $f : X \to X$ uniformly continuous, $x \in X$, $n \in \mathbb{N}$, $\epsilon > 0$

$$B(x, n, \epsilon) := \{y \in X : \max_{0 \le i \le n-1} \mathrm{dist}(f^i x, f^i y) < \epsilon\}.$$

For $K \subseteq X$, $F \subseteq X$ (n, ϵ)-*spans* K if $K \subseteq \bigcup_{x \in F} B(x, n, \varepsilon)$.
If K is compact, $N(n, \epsilon, K)$ is the smallest cardinality of any (n, ϵ)-spans sets for K.

Definition (Topological entropy)

$$\mathrm{h_{top}}(f; K) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon, K) \ge 0$$

The *topological entropy of f* is

$$\mathrm{h_{top}}(f) := \sup_{K \subset X \text{ compact}} \{\mathrm{h_{top}}(f; K)\}.$$

$f : X \to X$ continuous map on compact metric space X.
$\mathcal{M}_1(f)$ - set of f-invariant probability measures.

Variational Principle

$$\mathrm{h}_{\mathrm{top}}(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}_1(f)\}.$$

Definition (Measure of maximal entropy)

$\mu \in \mathcal{M}_1(f)$ is a measure of *maximal entropy (mme)* if

$$h_\mu(f) = \mathrm{h}_{\mathrm{top}}(f).$$

If there exists a unique mme, then f is intrinsically ergodic.

Motivation

For compact surfaces, Nielsen-Thurston classification:

Given an (orientation preserving) diffeomorphism f there exists g homotopic to it satisfying one of the following

- g^p is the identity for some $p \in \mathbb{N}$, or
- g is pseudo-Anosov, or
- g leaves invariant some finite set of closed simple curves.

- Periodic maps have zero entropy.

- (Fathi-Shub, 2012) $f : S \to S$ diffeomorphism in the isotopy class of a pseudo-Anosov $A : S \to S$, then $\mathrm{h_{top}}(f) \geq \mathrm{h_{top}}(A)$.

- The third case is reducible to the other cases.

Questions

- Can we characterize the minimizers in $[f]$?
- Can we give sufficient conditions for a map $g \in [f]$ to be a minimizer of the topological entropy?

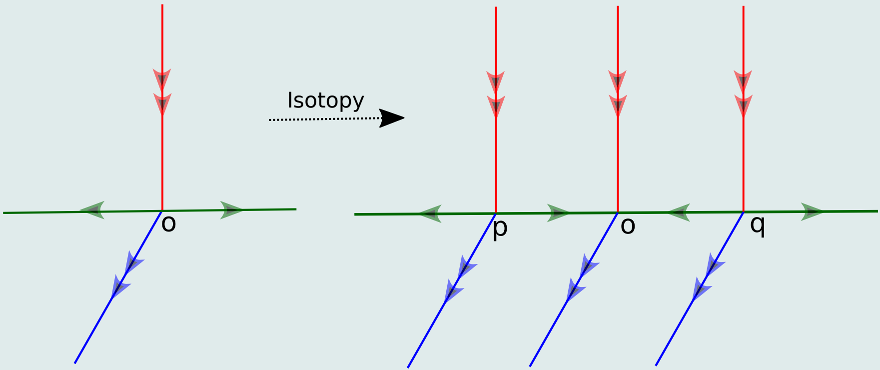
Derived from Anosov (DA)

Definition (DA)

$f : M \to M$ $\mathcal{C}^1$-diffeo is called *Derived from Anosov* if it is isotopic to an Anosov diffeomorphism.

If $M = \mathbb{T}^d$, then f is isotopic to its action in the homology $A : H_1(\mathbb{T}^d) \to H_1(\mathbb{T}^d)$. We call A the linear part of f.

1D – Central bundle



- Hertz-Hertz-Tahzibi-Ures, 2011.
- Buzzi-Fisher-Sambarino-Vásquez, 2012.
- Ures, 2012.

$f : \mathbb{T}^d \to \mathbb{T}^d$ DA with 1D center bundle, then

$$\mathrm{h}_{\mathrm{top}}(f) = \mathrm{h}_{\mathrm{top}}(A).$$

Moreover, f is intrinsically ergodic.

$dim(E^c) \geq 2$

What happens for higher center bundle dimension ?

- Newhouse-Young, 1983.
- Díaz-Fisher-Pacífico-Vieitez, 2012.
- Carrasco-Lizana-Pujals-Vásquez, 2021.
- Álvarez-Sánchez-Varão, 2021.

DA with $dim(E^c) \geq 2$

Thm. A (Carrasco-L.-Pujals-Vásquez'21)

$f : \mathbb{T}^d \to \mathbb{T}^d$ a DA partially hyperbolic diffeomorphism. Assume further that

1. the lifts of $\mathcal{F}^{cs}, \mathcal{F}^u$ to $\mathbb{R}^d$ have GPS, and likewise for $\mathcal{F}^s, \mathcal{F}^{cu}$;
2. E^c is strongly simple.

Then $\mathrm{h}_{\mathrm{top}}(f) = \mathrm{h}_{\mathrm{top}}(A)$.

If furthermore E^c is dominated, then the same is true for $\mathcal{C}^1$ small perturbations g of f, provided that g has simple center bundle.

Global Product Structure

We assume that the pairs of foliations $\tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^{cu}$ and $\tilde{\mathcal{F}}^u, \tilde{\mathcal{F}}^{cs}$ have GPS: for $x, y \in \mathbb{R}^d$ we denote

$$\langle x, y \rangle_{csu} = \tilde{W}^{cs}(x) \cap \tilde{W}^u(y)$$
$$\langle x, y \rangle_{cus} = \tilde{W}^{cu}(x) \cap \tilde{W}^s(y).$$

E^c is strongly simple

E^c is *simple* if

a) $E^c = E^1 \oplus \cdots \oplus E^\ell$ with $\dim E^i = 1$, $\forall\, i = 1, \dots, \ell$.

b) $\forall\, S \subset \{1, \cdots, \ell\}$, $E^S := \oplus_{i \in S} E^i$ integrates to an f-invariant foliation $\mathcal{F}^S$.
Furthermore, there is compatibility in the sense:
$S \subset S' \Rightarrow \mathcal{F}^S$ sub-foliates $\mathcal{F}^{S'}$.

E^c is *strongly simple* if it is simple and furthermore

c) For every i, the lifts of $\mathcal{F}^i := \mathcal{F}^{\{i\}}, \mathcal{F}^{\{1,\cdots,\hat{i},\cdots,\ell\}}$ to the universal covering of M have GPS inside each leaf of the lift of $\mathcal{F}^c$.

Thm. B (Carrasco-L.-Pujals-Vásquez'21)

There exist $g : \mathbb{T}^4 \to \mathbb{T}^4$ PH-DA with linear part A, $\mathcal{U}$ $\mathcal{C}^1$-neighborhood of g and $c > 0$ such that $\forall\, g' \in \mathcal{U}$ it holds

1. the lifts of $\mathcal{F}^{cs}_{g'}, \mathcal{F}^{u}_{g'}$ to $\mathbb{R}^d$ have GPS, and likewise for $\mathcal{F}^{s}_{g'}, \mathcal{F}^{cu}_{g'}$;
2. $\mathrm{h_{top}}(g') \geq \mathrm{h_{top}}(A) + c$;
3. g' is transitive.

Special class of DA [Carvalho'93]

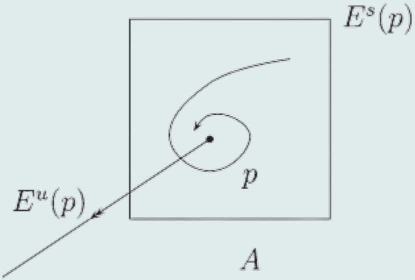
$A \in SL(3,\mathbb{Z})$ with eigenvalues $\lambda^u_A \in \mathbb{R}$, and $\lambda^s_A, \overline{\lambda^s_A} \in \mathbb{C}$, s.t.

$$0 < |\lambda^s_A| = |\overline{\lambda^s_A}| < 1 < |\lambda^u_A|.$$

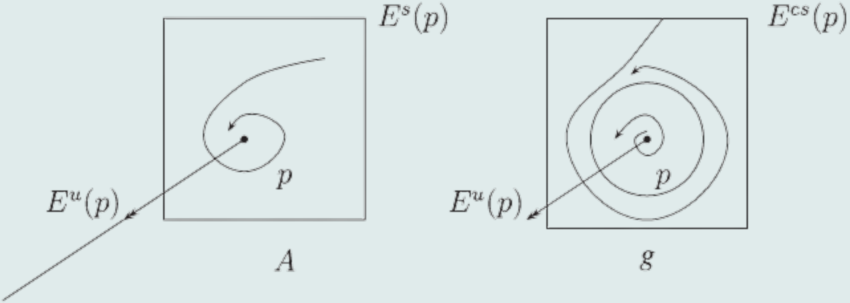
$f_A : \mathbb{T}^3 \to \mathbb{T}^3$ transitive linear Anosov diff. induced by A with

$$T\mathbb{T}^3 = E^s \oplus E^u,$$

$dim(E^s) = 2$, $dim(E^u) = 1$.



Let p a fixed point of f_A and fix $r_0 > 0$ and $\delta > 0$ sufficiently small such that $r_0 > 4\delta$.



f_A is deformed through an isotopy $\{f_t\}_{t\in[-1,1]}$ supported in $B(p, \frac{3r_0}{4})$ and satisfying:

- f_t is C^1-close to f_A, $\forall t < 0$.
- $\mathcal{F}^s_A$ is f_t-invariant and $f_t(p) = p$ for $t \in [-1, 1]$.
- The isotopy changes the stability index of p through a Hopf bifurcation in $\mathcal{F}^s_A(p)$ at $t = 0$, turning p a source for $t > 0$.

- There are $0 < |\lambda_A^s| \leq \lambda_t < 1 < \sigma_t < \beta_t \leq |\lambda_u|$ s.t for every unit vectors $v^{cs} \in C_t^{cs}(x)$ and $v^{uu} \in C_t^{uu}(x)$
  - $||D_x f_t(v^{cs})|| \leq \sigma_t, \quad x \in B(p, \frac{r_0}{2}),$
  - $||D_x f_t(v^{cs})|| \leq \lambda_t, \quad x \in \mathbb{T}^3 \setminus B(p, \frac{r_0}{2}),$
  - $\beta_t \leq ||D_x f_t(v^{uu})|| \leq |\lambda_u|, \quad x \in \mathbb{T}^3.$
- There exist $0 < \kappa_t < 1$ and a neighbourhood $V_t(p)$ contained in $B(p, \frac{r_0}{2}) \cap W_t^u(p)$ s.t
  - $J^c = |det(D_x f_t |_{T_x \mathcal{F}^c(x)})| \leq \kappa_t, \quad \forall x \in \mathbb{T}^3 \setminus V_t(p).$

Proposition (Properties of the DA)

For each $t \in (0,1]$ and $g := f_t : \mathbb{T}^3 \to \mathbb{T}^3$ as above holds:

1. $\exists h_g$ semi-conjugation between g and f_A such that $d_{C^0}(h_g, Id) < \delta$.
2. g is partially hyperbolic, dynamical coherence with indecomposable 2-dimensional central subbundle and minimal central foliation.
3. g has global product structure(GPS).
4. All equivalence classes $h_g^{-1}(h_g(x))$ are contained in a single center leaf.

Main result

Main Theorem (L.-Parra-Vásquez'23)

$g : \mathbb{T}^3 \to \mathbb{T}^3$ (defined as above) has a unique measure of maximal entropy. This measure is ergodic and hyperbolic. Moreover, g preserve the topological entropy of f_A.

Strategy for existence of mme

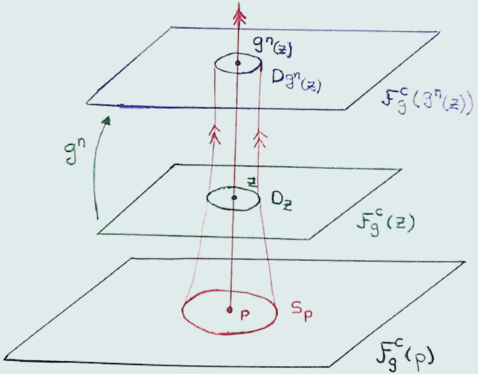
If $g, f : X \to X$ are continuous and h a semi-conjugation between g and f. Assuming

- $\exists \mu \in \mathcal{M}_1(f)$ a *mme* for f, and
- $\exists \nu \in \mathcal{M}_1(g)$ such that $h_* \nu = \mu$.

follows that

1. $\mathrm{h_{top}}(g, h^{-1}(x)) = 0, \ \forall x \in X \Rightarrow \nu$ is a *mme* for g.
2. $\mu(\{h(y) : h^{-1}(h(y)) = \{y\}\}) = 1 \Rightarrow \nu$ is the unique mme for g so that $h_*(\nu) = \mu$.

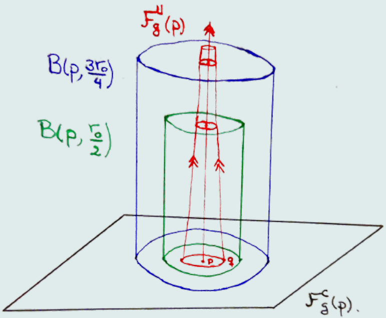
- $h_g^{-1}(h_g(z)) = \{y \in \mathbb{T}^3 : d(g^n(y); g^n(z)) \leq \delta, \forall n \in \mathbb{Z}\} = D_z$, $\forall z \in \mathcal{F}_g^u(p)$.



- $W_g^u(p)$ is an open dense set. Moreover,

$$\mathbb{T}^3 \setminus (W_g^u(p) \cup \bigcup_{q \in S_p} \mathcal{F}_g^u(q)) \neq \emptyset.$$

- For every $x_0 \in \mathbb{T}^3 \setminus (W_g^u(p) \cup \bigcup_{q \in S_p} \mathcal{F}_g^u(q))$ given by **Bonatti-Viana** has trivial equivalence class. Moreover, if $z \in \mathcal{F}_g^u(x_0)$, then $h_g^{-1}(h_g(z))$ is also trivial.



- $\mathrm{h}_{\mathrm{top}}(g, h_g^{-1}(h_g(z))) = 0, \ \forall z \in W_g^u(p) \cup \bigcup_{q \in S_p} \mathcal{F}_g^u(q).$

- There exists $B \subset \mathbb{T}^3$ with total m-Lebesgue measure such that every $x \in B$ has trivial equivalence class, that is,

$$m(\{x \in \mathbb{T}^3 : \# h_g^{-1}(x) = 1\}) = 1.$$

- Let $f : M \to M$ PH diff. Suppose that there exist an open $\mathcal{U} \subset M$ and $0 < \lambda_s \leq \lambda < 1 < \beta \leq \lambda_u$ s.t.

$$\max_{x \in \mathcal{U}} \{||D_x f \mid_{E^c_x} ||\} \leq \beta, \; \max_{x \in M \setminus \mathcal{U}} \{||D_x f \mid_{E^c_x} ||\} \leq \lambda.$$

  Every f-invariant ergodic measure μ such that $\mu(\mathcal{U}) \ll 1$ is hyperbolic.

  Moreover, $\lambda^c(x) < 0$ almost every point $x \in M$. ∎

Obrigada