Pullback attractors for semilinear non-autonomous problem

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[1] With Everaldo Bonotto (ICMC) and Eric Santiago (UFSCar)

The problem

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geqslant 3$, which boundary $\partial\Omega$ is sufficiently regular. We consider the following initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}} u_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}} v_t - a_\epsilon(t)(-\Delta)^{\frac{1}{2}} u_t = 0, \end{cases} \tag{1}$$

$(x,t) \in \Omega \times (\tau, \infty)$, where η is a positive constant, subject to boundary conditions

$$u = v = 0,\ (x,t) \in \partial\Omega \times (\tau, \infty), \tag{2}$$

and initial conditions

$$\begin{aligned} u(\tau, x) = u_0(x),\ u_t(\tau, x) = u_1(x), \\ v(\tau, x) = v_0(x), v_t(\tau, x) = v_1(x),\ x \in \Omega,\ \tau \in \mathbb{R}. \end{aligned} \tag{3}$$

Assume that the function $a_\epsilon \colon \mathbb{R} \to (0, \infty)$ is continuously differentiable in $\mathbb{R}$ and satisfies the following condition:

$$0 < a_0 \le a_\epsilon(t) \le a_1, \tag{4}$$

for all $\epsilon \in [0,1]$ and $t \in \mathbb{R}$, with positive constants a_0 and a_1, and we also assume that the first derivative of a_ϵ is uniformly bounded in t and ϵ, that is, there exists a constant $b_0 > 0$ such that

$$|a'_\epsilon(t)| \le b_0 \quad \text{for all} \quad t \in \mathbb{R},\ \epsilon \in [0,1]. \tag{5}$$

Furthermore, we assume that a_ϵ is (β, C)-Hölder continuous, for each $\epsilon \in [0,1]$; that is,

$$|a_\epsilon(t) - a_\epsilon(s)| \le C|t-s|^\beta \tag{6}$$

for all $t, s \in \mathbb{R}$ and $\epsilon \in [0,1]$.

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for all $t, s \in \mathbb{R}$ and $\epsilon \in [0, 1]$. Concerning the nonlinearity f, we assume that $f \in C^1(\mathbb{R})$ and it satisfies the dissipativeness condition

$$\limsup_{|s| \to \infty} \frac{f(s)}{s} \leq 0, \tag{7}$$

and also satisfies the subcritical growth condition given by

$$|f'(s)| \leq c(1 + |s|^{\rho - 1}), \tag{8}$$

for all $s \in \mathbb{R}$, where $1 < \rho < \frac{n}{n-2}$, with $n \geq 3$, and $c > 0$ is a constant.

In the case that $a_\epsilon(t) \equiv a$, the system (1) represents the autonomous version of the Klein-Gordon-Zakharov system. Within the autonomous case, if $n = 3$ then the Klein-Gordon-Zakharov system arises to describe the interaction of a Langmuir wave (Plasma oscillations, are rapid oscillations of the electron density in conducting media such as plasmas or metals in the ultraviolet region) and acoustic wave in a plasma.

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For more details see [**Bonotto, Nascimento and Santiago**, Long-time behaviour for a non-autonomous Klein-Gordon-Zakharov system, Journal of Mathematical Analysis and Applications, **506** (2022), 125670.

Suppose that we have a non-autonomous differential equations in a Banach space X

$$\frac{du}{dt} = f(t,u), \qquad u(s) = u_0,$$

with a unique solution $u(t,s,u_0)$. Note that the initial time has a very important role because we have an explicit dependence on time of f. This time dependence may appear in external force, in the operator, in both at the same time or even on the boundary conditions.

Basic Concepts

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In general, a non-autonomous system shows two different important dynamics without relation between them:

1. forward dynamic: the behavior when final time goes to infinity:

$$\lim_{t\to\infty} u(t, s, u_0).$$

2. pullback dynamics: the behavior when the initial time goes to minus infinity:

$$\lim_{s\to-\infty} u(t, s, u_0).$$

AUTONOMOUS	$\longleftrightarrow$	NONAUTONOMOUS
Semigroup $T(t)$		Process $U(t,s)$
exponential decay $\lVert T(t) \rVert \leqslant e^{-\beta t}$		exponential stability $\lVert U(t,s) \rVert \leqslant e^{-\beta(t-s)}$
Invariance $T(t)A = A$		Invariance $U(t,s)A(s) = A(t)$
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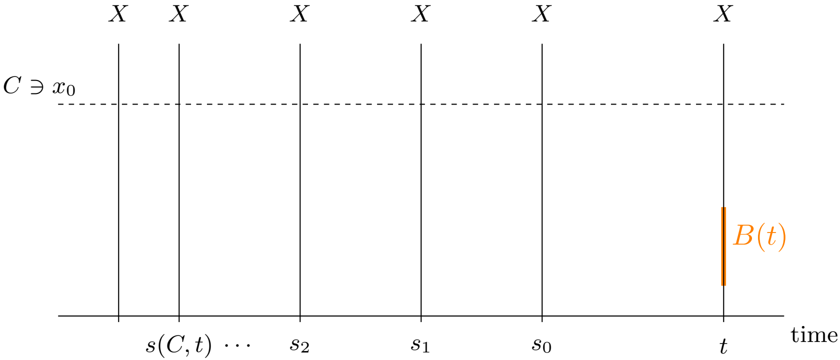
For more details see [**Carvalho, Langa and Robinson**, Attractors for infinite-dimensional non-autonomous semantical systems, Springer, 2012.]

Pullback attraction



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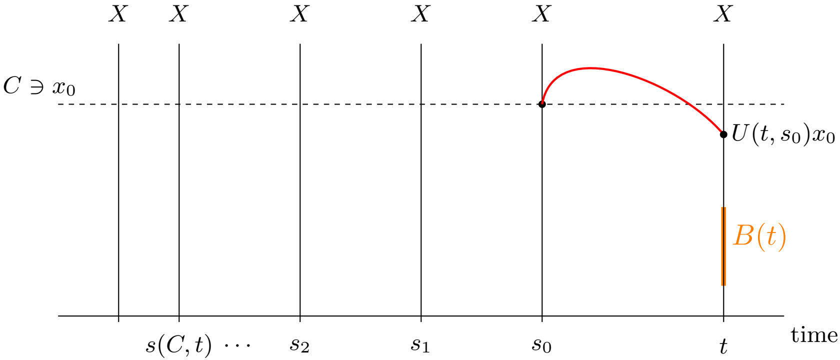

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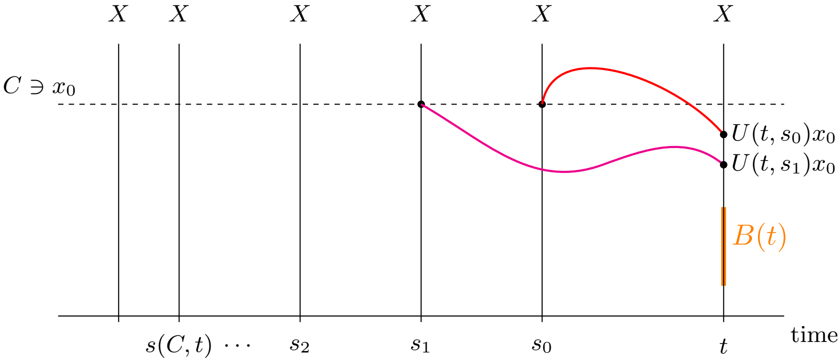

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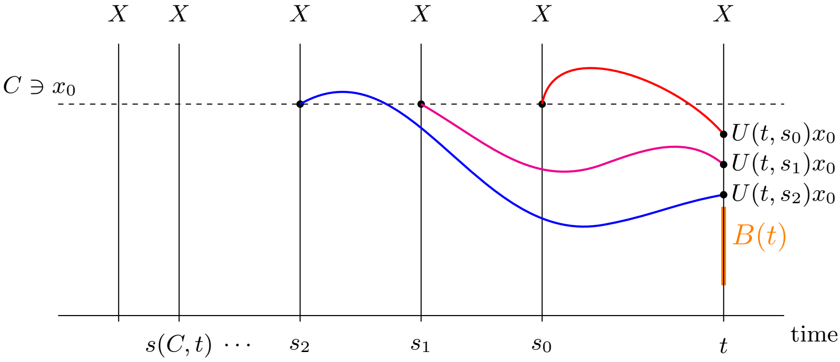

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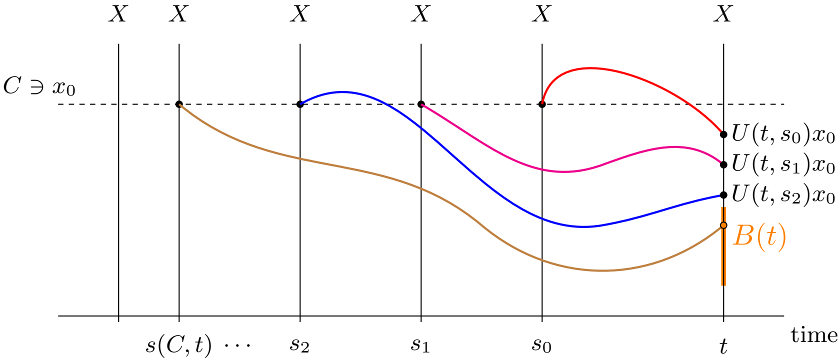

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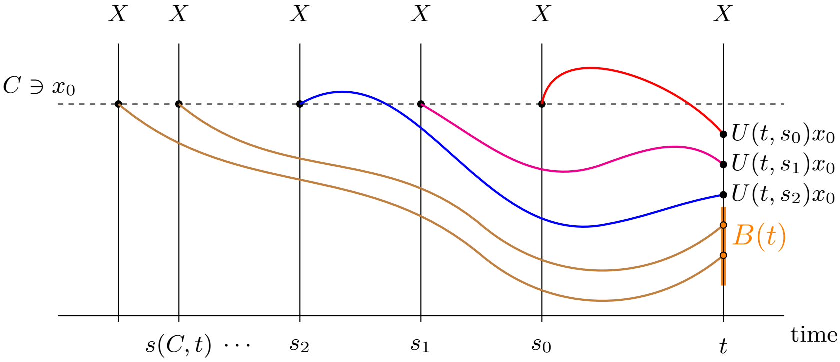

Singularly non-autonomous abstract problem

Here, $L(\mathcal{Z})$ will denote the space of linear and bounded operators defined in a Banach space $\mathcal{Z}$. Let $\mathcal{A}(t)$, $t \in \mathbb{R}$, be a family of unbounded closed linear operators defined on a fixed dense subspace D of $\mathcal{Z}$.

Consider the singularly non-autonomous parabolic problem

$$\begin{cases} \dfrac{du}{dt} + \mathcal{A}(t)u = 0, \ t > \tau, \\ u(\tau) = I. \end{cases} \tag{9}$$

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We assume

(a) The operator $\mathcal{A}(t) : D \subset \mathcal{Z} \to \mathcal{Z}$ is a closed densely defined operator (the domain D is fixed) and there is a constant $C > 0$ (independent of $t \in \mathbb{R}$) such that

$$\|(\lambda I + \mathcal{A}(t))^{-1}\|_{L(\mathcal{Z})} \leqslant \frac{C}{|\lambda| + 1}; \text{ for all } \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geqslant 0.$$

To express this fact we will say that the family $\mathcal{A}(t)$ is uniformly sectorial.

(*b*) There are constants $C > 0$ and $\epsilon_0 > 0$ such that, for any $t, \tau, s \in \mathbb{R}$,

$$\|[\mathcal{A}(t) - \mathcal{A}(\tau)]\mathcal{A}^{-1}(s)\|_{L(\mathcal{Z})} \leqslant C(t-\tau)^{\epsilon_0}, \quad \epsilon_0 \in (0,1].$$

To express this fact we will say that the map $\mathbb{R} \ni t \mapsto \mathcal{A}(t)$ is uniformly Hölder continuous.

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Denote by $\mathcal{A}_0$ the operator $\mathcal{A}(t_0)$ for some $t_0 \in \mathbb{R}$ fixed. If $\mathcal{Z}^\alpha$ denotes the domain of $\mathcal{A}_0^\alpha$, $\alpha > 0$, with the graph norm and $\mathcal{Z}^0 := \mathcal{Z}$, denote by $\{\mathcal{Z}^\alpha; \alpha \geqslant 0\}$ the fractional power scale associated with $\mathcal{A}_0$ (see **Henry** [Springer, 1981] and **Amann** [Birkhäuser Verlag, Basel, 1995].

From (a), $-\mathcal{A}(t)$ is the infinitesimal generator of an analytic semigroup $\{e^{-\tau\mathcal{A}(t)} \in L(\mathcal{Z}) : \tau \geqslant 0\}$. Using this and the fact that $0 \in \rho(\mathcal{A}(t))$, it follows that

$$\|e^{-\tau\mathcal{A}(t)}\|_{L(\mathcal{Z})} \leqslant Ce^{-\delta\tau},\ \delta > 0, \tau \geqslant 0,\ t \in \mathbb{R},$$

and

$$\|\mathcal{A}(t)e^{-\tau\mathcal{A}(t)}\|_{L(\mathcal{Z})} \leqslant C\tau^{-1}e^{-\delta\tau},\ \delta > 0,\ \tau > 0,\ t \in \mathbb{R}.$$

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It follows from (b) that $\|\mathcal{A}(t)\mathcal{A}^{-1}(\tau)\|_{L(\mathcal{Z})} \leqslant C$, $\forall\, t, \tau \in I$, for all $I \subset \mathbb{R}$ bounded. Also, the semigroup $e^{-\tau\mathcal{A}(t)}$ generated by $-\mathcal{A}(t)$ satisfies the following estimate

$$\|e^{-\tau\mathcal{A}(t)}\|_{L(\mathcal{Z}^\beta,\mathcal{Z}^\alpha)} \leqslant M\tau^{\beta-\alpha},$$

where $0 \leqslant \beta \leqslant \alpha < 1 + \epsilon_0$.

Next we recall the definition of a linear evolution process associated with a family of operators $\{\mathcal{A}(t) : t \in \mathbb{R}\}$.

Definition 1

A family $\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\} \subset L(\mathcal{Z})$ satisfying

1) $L(\tau,\tau) = I$,
2) $L(t,\sigma)L(\sigma,\tau) = L(t,\tau)$, for any $t \geqslant \sigma \geqslant \tau$,
3) $\mathcal{P} \times \mathcal{Z} \ni ((t,\tau), u_0) \mapsto L(t,\tau)v_0 \in \mathcal{Z}$ is continuous,

where $\mathcal{P} = \{(t,\tau) \in \mathbb{R}^2 : t \geqslant \tau\}$, is called a *linear evolution process* (*process* for short) or *family of evolution operators*.

If the operator $\mathcal{A}(t)$ is uniformly sectorial and uniformly Hölder continuous, then there exists a linear evolution process $\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ associated with $\mathcal{A}(t)$, which is given by

$$L(t,\tau) = e^{-(t-\tau)\mathcal{A}(\tau)} + \int_{\tau}^{t} L(t,s)[\mathcal{A}(\tau) - \mathcal{A}(s)]e^{-(s-\tau)\mathcal{A}(\tau)}ds,$$

that is solution of (9).

For more details see **Carvalho and Nascimento** [DCDS, 2009].

We consider the singularly non-autonomous abstract parabolic problem

$$\begin{cases} \dfrac{du}{dt} + \mathcal{A}(t)u = g(t,u), \ t > \tau, \\ u(\tau) = u_0 \in D, \end{cases} \tag{10}$$

where the operator $\mathcal{A}(t)$ is uniformly sectorial and uniformly Hölder continuous and the nonlinearity g satisfies conditions which will be specified later.

Definition 2

Let $g : \mathbb{R} \times X^\alpha \to X^\beta$, $\alpha \in [\beta, \beta+1)$ be a continuous function. We say that a function u is a (*local*) *solution* of (10) starting in $u_0 \in X^\alpha$, if $u \in C([\tau, \tau+t_0], X^\alpha) \cap C^1((\tau, \tau+t_0], X^\alpha)$, $u(\tau) = u_0$, $u(t) \in D(\mathcal{A}(t))$ for all $t \in (\tau, \tau+t_0]$ and (10) is satisfied for all $t \in (\tau, \tau+t_0)$.

Now we state the following abstract local well-posedness result.

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Now we state the following abstract local well-posedness result.

Theorem 3 (Caraballo et al. Nonlinear Analysis (2010))

Suppose that the family of operators $\mathcal{A}(t)$ is uniformly sectorial and uniformly Hölder continuous in X^β. If $g : \mathbb{R} \times X^\alpha \to X^\beta$, $\alpha \in [\beta, \beta + 1)$, is a Lipschitz continuous map in bounded subsets of X^α, then, given $r > 0$, there is a time $t_0 > 0$ such that for all $u_0 \in B_{X^\alpha}(0; r)$ there exists a unique solution $u(\cdot, \tau, u_0) \in C([\tau, \tau + t_0], X^\alpha) \cap C^1((\tau, \tau + t_0], X^\alpha)$ of the problem (10) starting in $u_0 \in X^\alpha$. Moreover, such solutions are continuous with respect the initial data in $B_{X^\alpha}(0; r)$.

We start remembering the definition of Hausdorff semi-distance between two subsets A and B of a metric space (X, d):

$$\operatorname{dist}_H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

Basic definitions and existence results

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$$\mathrm{dist}_H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

Next we present several definitions about theory of pullback attractors.

Definition 4

Let $\{S(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ be an evolution process in a metric space X. A set $B(t) \subset X$ pullback attracts a set C at time t under $\{S(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ if

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(S(t, \tau)C, B(t)) = 0,$$

where $S(t, \tau)C := \{S(t, \tau)x \in X : x \in C\}$.

Definition 5

We say that an evolution process $\{S(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ in X is pullback strongly bounded if, for each $t \in \mathbb{R}$ and each bounded subset B of X,

$$\bigcup_{\tau \leqslant t} \bigcup_{s \leqslant \tau} S(\tau, s)B$$

is bounded.

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Definition 6

An evolution process $\{S(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ in X is pullback asymptotically compact if, for each $t \in \mathbb{R}$, each sequence $\{\tau_n\}$ in $(-\infty, t]$ with $\tau_n \to -\infty$ as $n \to \infty$ and each bounded sequence $\{x_n\}$ in X such that $\{S(t,\tau_n)x_n\} \subset X$ is bounded, the sequence $\{S(t,\tau_n)x_n\}$ is relatively compact in X.

Definition 7

A family $\{\mathbb{A}(t) \colon t \in \mathbb{R}\}$ of compact subsets of X is a pullback attractor for an evolution process $\{S(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$ if the following conditions hold:

(i) $\{\mathbb{A}(t) \colon t \in \mathbb{R}\}$ is invariant, that is, $S(t,\tau)\mathbb{A}(\tau) = \mathbb{A}(t)$ for all $t \geq \tau$,

(ii) $\{\mathbb{A}(t) \colon t \in \mathbb{R}\}$ pullback attracts bounded subsets of X, that is,

$$\lim_{\tau \to -\infty} \mathrm{d}_{\mathrm{X}}(S(t,\tau)B, \mathbb{A}(t)) = 0$$

for every $t \in \mathbb{R}$ and every bounded subset B of X, where $S(t,\tau)B = \{S(t,\tau)x \colon x \in B\}$ is the image of B under $\{S(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$, and

(iii) $\{\mathbb{A}(t) \colon t \in \mathbb{R}\}$ is the minimal family of closed sets satisfying property *(ii)*.

In applications, to prove that a process has a pullback attractor we use the Theorem below, proved in **Caraballo et al., [Nonlinear Anal. (2010)]** which gives a sufficient condition for existence of a pullback attractor.

Definition 8

An evolution process $\{S(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ in X is pullback strongly bounded dissipative if, for each $t \in \mathbb{R}$, there is a bounded subset $B(t)$ of X which pullback absorbs bounded subsets of X at time s for each $s \leqslant t$; that is, given a bounded subset B of X and $s \leqslant t$, there exists $\tau_0(s,B)$ such that $S(s,\tau)B \subset B(t)$, for all $\tau \leqslant \tau_0(s,B)$.

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Theorem 9 (CCLR, 2010)

If an evolution process $\{S(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ in the metric space X is **pullback strongly bounded dissipative** *and* **pullback asymptotically compact**, *then $\{S(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ has a pullback attractor $\{\mathbb{A}(t) : t \in \mathbb{R}\}$ with the property that $\bigcup\limits_{\tau \leqslant t} \mathbb{A}(\tau)$ is bounded for each $t \in \mathbb{R}$.*

Theorem 10 (CCLR, 2010)

Let $\{S(t,s) : t \geqslant s \in \mathbb{R}\}$ be a pullback strongly bounded evolution process such that $S(t,s) = L(t,s) + U(t,s)$, where there exist a non-increasing function $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, with $k(\sigma, r) \to 0$ when $\sigma \to \infty$, and for all $s \leqslant t$ and $x \in X$ with $\|x\| \leqslant r$, $\|L(t,s)x\| \leqslant k(t-s, r)$, and $U(t,s)$ is compact. Then, the family of evolution process $\{S(t,s) : t \geqslant s \in \mathbb{R}\}$ is pullback asymptotically compact.

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For more details see [**Carvalho, Langa and Robinson**, Attractors for infinite-dimensional non-autonomous semantical systems, Springer, 2012.]

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$(x,t) \in \Omega \times (\tau, \infty)$, where η is a positive constant, subject to boundary conditions

$$u = v = 0, \ (x,t) \in \partial\Omega \times (\tau, \infty), \tag{12}$$

and initial conditions

$$\begin{aligned} u(\tau, x) = u_0(x), \ u_t(\tau, x) = u_1(x), \\ v(\tau, x) = v_0(x), v_t(\tau, x) = v_1(x), \ x \in \Omega, \ \tau \in \mathbb{R}. \end{aligned} \tag{13}$$

In order to formulate the non-autonomous problem $(1)-(3)$ in a nonlinear evolution process setting, we introduce some notations. Let $X = L^2(\Omega)$ and denote by $A\colon D(A) \subset X \to X$ the negative Laplacian operator, that is, $Au = (-\Delta)u$ for all $u \in D(A)$, where $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Thus A is a positive self-adjoint operator in X with compact resolvent and, therefore, $-A$ generates a compact analytic semigroup on X. Following **Henry** [Springer, 1981]), A is a sectorial operator in X. Now, denote by X^α, $\alpha > 0$, the fractional power spaces associated with the operator A; that is, $X^\alpha = D(A^\alpha)$ endowed with the graph norm. With this notation, we have $X^{-\alpha} = (X^\alpha)'$ for all $\alpha > 0$, see **Amann** (Birkhäuser Verlag, 1995).

In this framework, the non-autonomous problem $(1)-(3)$ can be rewritten as an ordinary differential equation in the following abstract form

$$\begin{cases} W_t + \mathcal{A}(t)W = F(W), & t > \tau, \\ W(\tau) = W_0, & \tau \in \mathbb{R}, \end{cases} \tag{14}$$

where $W = W(t)$, for all $t \in \mathbb{R}$, and $W_0 = W(\tau)$ are respectively given by

$$W = \begin{bmatrix} u \\ u_t \\ v \\ v_t \end{bmatrix} \text{ and } W_0 = \begin{bmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \end{bmatrix},$$

and, for each $t \in \mathbb{R}$, the unbounded linear operator $\mathcal{A}(t)\colon D(\mathcal{A}(t)) \subset Y \to Y$ is defined by

$$\mathcal{A}(t)\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} -v \\ (A+I)u + \eta A^{\frac{1}{2}} v + a_{\epsilon}(t) A^{\frac{1}{2}} z \\ -z \\ -a_{\epsilon}(t) A^{\frac{1}{2}} v + Aw + \eta A^{\frac{1}{2}} z \end{bmatrix} \tag{15}$$

for each $\begin{bmatrix} u & v & w & z \end{bmatrix}^T$ in the domain $D(\mathcal{A}(t))$ defined by the space

$$D(\mathcal{A}(t)) = Y^1 = X^1 \times X^{\frac{1}{2}} \times X^1 \times X^{\frac{1}{2}}, \tag{16}$$

where

$$Y = Y_0 = X^{\frac{1}{2}} \times X \times X^{\frac{1}{2}} \times X$$

is the phase space of the problem $(1) - (3)$.

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is the phase space of the problem $(1)-(3)$. The nonlinearity F is given by

$$F(W) = \begin{bmatrix} 0 \\ f^e(u) \\ 0 \\ 0 \end{bmatrix}, \tag{17}$$

where $f^e(u)$ is the Nemitskiĭ operator associated with $f(u)$; that is,

$$f^e(u)(x) = f(u(x)), \quad \text{for all} \quad x \in \Omega.$$

Now, we observe that the norms

$$\|(x, y, z, w)\|_1 = \|x\|_{X^{\frac{1}{2}}} + \|y\|_X + \|z\|_{X^{\frac{1}{2}}} + \|w\|_X$$

and

$$\|(x, y, z, w)\|_2 = (\|x\|^2_{X^{\frac{1}{2}}} + \|y\|^2_X + \|z\|^2_{X^{\frac{1}{2}}} + \|w\|^2_X)^{\frac{1}{2}}$$

are equivalent in Y_0. In this way, we shall use the same notation $\|(x, y, z, w)\|_{Y_0}$ for both norms and the choice will be as convenient.

In the abstract form our problem was written as

$$W_t + \mathcal{A}(t)W = F(W),$$

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$$\begin{aligned}\mathcal{E}(t) = \frac{1}{2}\|u(t)\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|u(t)\|^2_{X} + \frac{1}{2}\|u_t(t)\|^2_{X} + \frac{1}{2}\|v(t)\|^2_{X^{\frac{1}{2}}} \\ + \frac{1}{2}\|v_t(t)\|^2_{X} - \int_\Omega \int_0^u f(s)dsdx\end{aligned}$$

that is naturally defined in Y and we will show that $\mathcal{E}(t)$ decays along the solutions,

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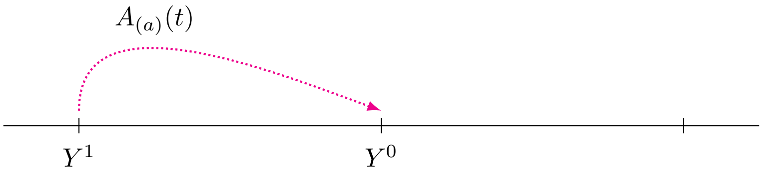
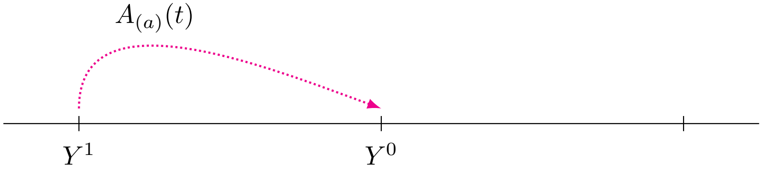
that is naturally defined in Y and we will show that $\mathcal{E}(t)$ decays along the solutions, and then we would like to write our problem in the form

$$\mathcal{A}(t) : Y \to Y_{-1}$$

for some appropriate space Y_{-1} such that $Y \subset Y_{-1}$ and

$$F(\cdot) : Y \to Y_{\alpha-1} \hookrightarrow Y_{-1}, \quad (0 < \alpha < 1).$$


$A_{(a)}(t)$
Y^1
Y^0



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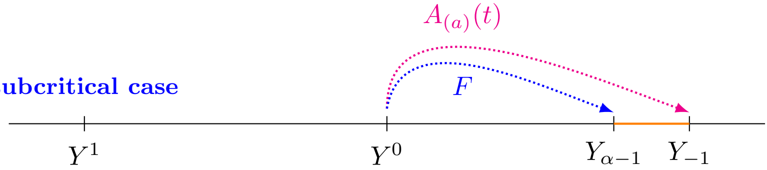

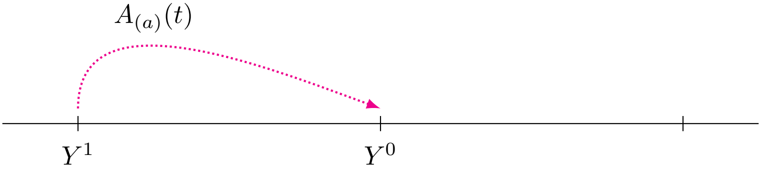

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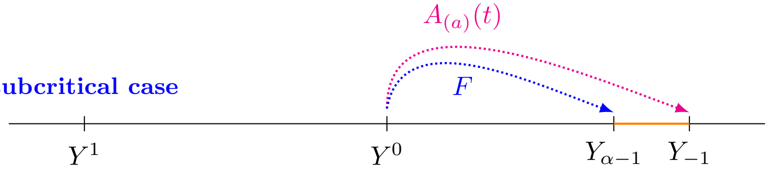

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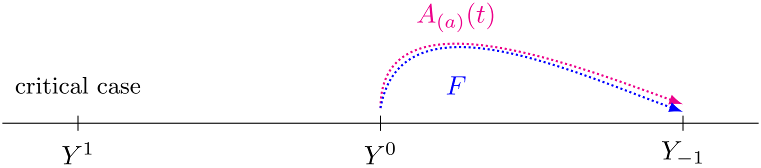

Main results

Theorem 11 (Well-Posedness)

Let $f \in C^1(\mathbb{R})$ be a function satisfying (7)-(8), assume conditions (4)-(6) hold and let $F\colon Y_0 \to Y_{\alpha-1} \subset Y_{-1}$ be defined in (17). Then for any initial data $W_0 \in Y_0$ the problem (14) has a unique global solution $W(t)$ such that

$$W(t) \in C([\tau, \infty), Y_0).$$

Moreover, such solutions are continuous with respect to the initial data on Y_0. Here, $Y_{-1} = X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{1}{2}}$.

Theorem 12 (Pullback Attractors)

Under the conditions of Theorem 11, the problem $(1)-(3)$ *has a pullback attractor* $\{\mathbb{A}(t) : t \in \mathbb{R}\}$ *in* Y_0 *and*

$$\bigcup_{t\in\mathbb{R}} \mathbb{A}(t) \subset Y_0$$

is bounded.

Consider the linear problem associated with (1)-(3), in this case we consider the singularly non-autonomous linear parabolic problem

$$\begin{cases} w_t + \mathcal{A}(t)w = 0, \ t > \tau, \\ w(\tau) = I, \end{cases} . \tag{18}$$

It is not difficult to see that $\det(\mathcal{A}(t)) = A(A + I)$, and therefore that $0 \in \rho(\mathcal{A}(t))$, for all $t \in \mathbb{R}$.

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It is not difficult to see that $\det(\mathcal{A}(t)) = A(A+I)$, and therefore that $0 \in \rho(\mathcal{A}(t))$, for all $t \in \mathbb{R}$. Moreover, for each $t \in \mathbb{R}$, the operator $\mathcal{A}^{-1}(t)\colon Y_0 \to Y_0$ is defined by

$$\mathcal{A}^{-1}(t) \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} \eta A^{\frac{1}{2}}(A+I)^{-1} & (A+I)^{-1} & a_\epsilon(t)A^{\frac{1}{2}}(A+I)^{-1} & 0 \\ -I & 0 & 0 & 0 \\ -a_\epsilon(t)A^{-\frac{1}{2}} & 0 & \eta A^{-\frac{1}{2}} & A^{-1} \\ 0 & 0 & -I & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}. \tag{19}$$

Proposition 13

If Y_{-1} *denotes the extrapolation space of* $Y_0 = X^{\frac{1}{2}} \times X \times X^{\frac{1}{2}} \times X$ *generated by the operator* $\mathcal{A}^{-1}(t)$*, then*

$$Y_{-1} = X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{1}{2}}.$$

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If Y_{-1} denotes the extrapolation space of $Y_0 = X^{\frac{1}{2}} \times X \times X^{\frac{1}{2}} \times X$ generated by the operator $\mathcal{A}^{-1}(t)$, then

$$Y_{-1} = X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{1}{2}}.$$

Recall that the extrapolation space Y_{-1} is the completion of the normed space $(Y, \|\mathcal{A}^{-1}(t) \cdot \|_Y)$.

Proposition 14

The family of operators $\{\mathcal{A}(t) : t \in \mathbb{R}\}$, defined in $(15)-(16)$, is uniformly Hölder continuous in Y_{-1}.

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Proposition 14

The family of operators $\{\mathcal{A}(t) : t \in \mathbb{R}\}$, defined in $(15) - (16)$, is uniformly Hölder continuous in Y_{-1}.

Proof: Using (6), this result follows immediately from (15) and (16). □

The next step is to show the analyticity of the semigroup $\{e^{-\tau\mathcal{A}(t)} : \tau \geq 0\}$.

Theorem 15

The semigroup $\{e^{-\tau\mathcal{A}(t)} : \tau \geq 0\}$, generated by $-\mathcal{A}(t)$, is analytic for each $t \in \mathbb{R}$.

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Theorem 15

The semigroup $\{e^{-\tau\mathcal{A}(t)} : \tau \geq 0\}$, generated by $-\mathcal{A}(t)$, is analytic for each $t \in \mathbb{R}$.

For the proof see [Bonotto, Nascimento and Santiago, JMAA (2022)].

Theorem 16

The operators $\mathcal{A}(t)$ are uniformly sectorial and the map $\mathbb{R} \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(Y_{-1})$ is uniformly Hölder continuous. Then, for each functional parameter a, there exist a process

$$\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$$

(or simply $L(t,\tau)$) associated with the operator $\mathcal{A}(t)$, that is solution of the linear problem associated with (1)-(3).

Remark 17

We have the following description of the fractional power scale for the operator $\mathcal{A}(t)$, given as follows

$$Y_0 \hookrightarrow Y_{\alpha-1} \hookrightarrow Y_{-1}, \quad \text{for all} \quad 0 < \alpha < 1,$$

where

$$\begin{aligned} Y_{\alpha-1} = [Y_{-1}, Y_0]_\alpha &= [X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{1}{2}}, X^{\frac{1}{2}} \times X \times X^{\frac{1}{2}} \times X]_\alpha \\ &= [X, X^{\frac{1}{2}}]_\alpha \times [X^{-\frac{1}{2}}, X]_\alpha \times [X, X^{\frac{1}{2}}]_\alpha \times [X^{-\frac{1}{2}}, X]_\alpha \\ &= X^{\frac{\alpha}{2}} \times X^{\frac{\alpha-1}{2}} \times X^{\frac{\alpha}{2}} \times X^{\frac{\alpha-1}{2}}, \end{aligned}$$

where $[\cdot,\cdot]_\alpha$ denotes the complex interpolation functor (see **Triebel** *(1978)). The first equality follows from Proposition* **??** *(since $0 \in \rho(A_{(a)}(t))$) see* **Amann** *(Example 4.7.3 (b)]) and the second equality follows from Proposition 2 in* **Carvalho and Cholewa** *(Bull. Austral. Math. Soc., (2002).*

Existence of local solutions

Proposition 18 gives us sufficient conditions for $F\colon Y_0 \to Y_{\alpha-1}$ to be Lipschitz continuous in bounded subsets of Y_0.

Proposition 18

Assume that $1 < \rho < \frac{n+2(1-\alpha)}{n-2}$, *with* $\alpha \in (0,1)$. *Then the map* $F\colon Y_0 \to Y_{\alpha-1}$, *defined in* (17), *is Lipschitz continuous in bounded subsets of* Y_0.

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Corollary 19

Let $1 < \rho < \frac{n+2(1-\alpha)}{n-2}$, *with* $\alpha \in (0,1)$, $f \in C^1(\mathbb{R})$ *be a function satisfying* (7)-(8), *assume conditions* (4)-(6) *hold and let* $F\colon Y_0 \to Y_{\alpha-1}$ *be defined in* (17). *Then given* $r > 0$, *there exists a time* $t_0 = t_0(r) > 0$ *such that for all* $W_0 \in B_{Y_0}(0,r)$, *there exists a unique solution* $W\colon [\tau, \tau + t_0] \to Y_0$ *of the problem* (14) *starting in* W_0. *Moreover, such solutions are continuous with respect to the initial data in* $B_{Y_0}(0,r)$.

Proof of Theorem 11: By Corollary 19, the problem (1)-(3) has a local solution $(u(t), u_t(t), v(t), v_t(t))$ in Y_0 defined on some interval $[\tau, \tau + t_0]$.
We can show that

$$\frac{d}{dt}\mathcal{E}(t) = -\eta\|(-\Delta)^{\frac{1}{4}}u_t\|_X^2 - \eta\|(-\Delta)^{\frac{1}{4}}v_t\|_X^2 \tag{20}$$

for all $\tau < t \leq \tau + t_0$, where

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for all $\tau < t \leq \tau + t_0$, where

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\|u(t)\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|u(t)\|_X^2 + \frac{1}{2}\|u_t(t)\|_X^2 \\ &\quad + \frac{1}{2}\|v(t)\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|v_t(t)\|_X^2 - \int_\Omega \int_0^u f(s)\,ds\,dx\end{aligned} \tag{21}$$

is the total energy associated with the solution $(u(t), u_t(t), v(t), v_t(t))$ of the problem (1)-(3) in Y_0.

The identity (20) means that the map $t \mapsto \mathcal{E}(t)$ is monotone decreasing along solutions. Moreover, using the property $\mathcal{E}(t) \leq \mathcal{E}(\tau)$ for all $\tau \leq t \leq \tau + t_0$, we can obtain a priori estimate of the solution $(u(t), u_t(t), v(t), v_t(t))$ in Y_0. In fact, we obtain

$$\|u\|^2_{X^{\frac{1}{2}}} + \|u_t\|^2_X + \|v\|^2_{X^{\frac{1}{2}}} + \|v_t\|^2_X \leq 4\left(\mathcal{E}(\tau) + C_{\frac{\lambda_1}{4}}\right),$$

that is,

$$\|(u(t), u_t(t), v(t), v_t(t))\|^2_{Y_0} \leq 4\left(\mathcal{E}(\tau) + C_{\frac{\lambda_1}{4}}\right).$$

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that is,

$$\|(u(t), u_t(t), v(t), v_t(t))\|^2_{Y_0} \leq 4\left(\mathcal{E}(\tau) + C_{\frac{\lambda_1}{4}}\right).$$

This ensures that the problem $(1)-(3)$ has a global solution $W(t)$ in Y_0, which proves the result. □

Since the problem $(1)-(3)$ has a global solution $W(t)$ in Y_0, we can define an evolution process $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ in Y_0 by

$$S(t,\tau)W_0 = W(t), \quad t \geq \tau \in \mathbb{R}. \tag{22}$$

Since the problem $(1)-(3)$ has a global solution $W(t)$ in Y_0, we can define an evolution process $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ in Y_0 by

$$S(t,\tau)W_0 = W(t), \quad t \geq \tau \in \mathbb{R}. \tag{22}$$

According to **Carvalho and Nascimento** (DCDS-S 2009)

$$S(t,\tau)W_0 = L(t,\tau)W_0 + U(t,\tau)W_0, \quad t \geq \tau \in \mathbb{R}, \tag{23}$$

where $\{L(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is the linear evolution process in Y_0 associated with the homogeneous problem

$$\begin{cases} W_t + \mathcal{A}(t)W = 0, \ t > \tau, \\ W(\tau) = W_0, \ \tau \in \mathbb{R}, \end{cases} \tag{24}$$

and

$$U(t,\tau)W_0 = \int_\tau^t L(t,s)F(S(s,\tau)W_0)ds. \tag{25}$$

In this section, we prove the existence of the pullback attractor of the problem (1)-(3). To this end, we need to make a modification on the energy functional. More precisely, for $\gamma_1, \gamma_2 \in \mathbb{R}_+$, let us define $L_{\gamma_1,\gamma_2} : Y_0 \to \mathbb{R}$ by the map

$$\begin{aligned} L_{\gamma_1,\gamma_2}(\phi, \varphi, \psi, \Phi) = & \frac{1}{2}\|\phi\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|\phi\|^2_X + \frac{1}{2}\|\varphi\|^2_X + \frac{1}{2}\|\psi\|^2_{X^{\frac{1}{2}}} \\ & + \frac{1}{2}\|\Phi\|^2_X + \gamma_1\langle\phi, \varphi\rangle_X + \gamma_2\langle\psi, \Phi\rangle_X \\ & - \int_\Omega \int_0^\phi f(s)dsdx. \end{aligned} \tag{26}$$

Theorem 20

There exists $R > 0$ such that for any bounded subset $B \subset Y_0$ one can find $t_0(B) > 0$ satisfying

$$\|(u, u_t, v, v_t)\|^2_{Y_0} \leq R \quad \textit{for all} \quad t \geq \tau + t_0(B).$$

In particular, the evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ defined in (22) is pullback strongly bounded dissipative.

Theorem 20

There exists $R > 0$ such that for any bounded subset $B \subset Y_0$ one can find $t_0(B) > 0$ satisfying

$$\|(u, u_t, v, v_t)\|_{Y_0}^2 \leq R \quad \text{for all} \quad t \geq \tau + t_0(B).$$

In particular, the evolution process $\{S(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$ defined in (22) is pullback strongly bounded dissipative.

Next, we prove that the solutions of problem (14) are uniformly exponentially dominated when the initial data are in bounded subsets of Y_0.

Theorem 21

Let $B \subset Y_0$ be a bounded set. If $W \colon [\tau, \infty) \to Y_0$ is the global solution of (14) starting at $W_0 \in B$, then there are positive constants $\sigma = \sigma(B)$, $K_1 = K_1(B)$ and $K_2 = K_2(B)$ such that

$$\|W(t)\|_{Y_0}^2 \leq K_1 e^{-\sigma(t-\tau)} + K_2, \quad t \geq \tau.$$

Theorem 22

Let $B \subset Y_0$ be a bounded set and denote by $L\colon [\tau, \infty) \to Y_0$ the solution of the homogeneous problem (24) starting in $W_0 \in B$. Then there exist positive constants $K = K(B)$ and ζ such that

$$\|L(t)\|_{Y_0}^2 \leq K e^{-\zeta(t-\tau)}, \quad t \geq \tau.$$

Theorem 22

Let $B \subset Y_0$ be a bounded set and denote by $L\colon [\tau, \infty) \to Y_0$ the solution of the homogeneous problem (24) starting in $W_0 \in B$. Then there exist positive constants $K = K(B)$ and ζ such that

$$\|L(t)\|_{Y_0}^2 \leq K e^{-\zeta(t-\tau)}, \quad t \geq \tau.$$

Proposition 23

For each $t > \tau \in \mathbb{R}$, the evolution process $S(t,\tau)\colon Y_0 \to Y_0$ given in (22) is a compact map.

Proof of Theorem 12: Theorem 20 assures that the evolution process $S(t,\tau)\colon Y_0 \to Y_0$ given by (22) is pullback strongly bounded dissipative. Additionally, it follows by Proposition 23 that $S(t,\tau)\colon Y_0 \to Y_0$ is compact, and, consequently, it is pullback asymptotically compact. Now the result is a simple consequence of Theorem 9. $\square$

Obrigado.