Almost topological spaces and modal axioms

Hércules de Araujo Feitosa
hercules.feitosa@unesp.br

UNESP - College of Sciences - Bauru
Department of Mathematics

VII Encontro da Pós-Graduação em Matemática da UFBA
November, 4 - 8, 2019

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Definitions

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- A set $A \in \mathcal{P}(E)$ is **closed set** when its complement relative to E is an open of (E, Ω).

Closure, Interior, Properties

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- Any intersection of closed sets of (E, Ω) is a closed set of (E, Ω).

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Topological spaces

Including conditions

- An almost topological space (E, Ω) is **0-closed** when it holds:
  (*zc*) $\overline{\emptyset} = \emptyset$.
- A **topological space** (E, Ω) is an almost topological space 0-closed such that it holds:
  (*ucl*) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

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- A **consequence operator** on E is a function $^{-} : \mathcal{P}(E) \to \mathcal{P}(E)$ such that, for every $A, B \in \mathcal{P}(E)$:
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- Of course, from (i) and (iii), the equality $\overline{\overline{A}} = \overline{A}$ holds, for every $A \subseteq E$.
- A **Tarski' space** (**Tarski's deductive system** or **closure space**) is a pair $(E, ^{-})$ such that E is a non-empty set and $^{-}$ is a consequence operator on E.

Tarski spaces

Closed and open sets

Closed and open sets

- Let $(E, ^{-})$ be a Tarski space. The set A is **closed** in $(E, ^{-})$ when $\overline{A} = A$, and A is **open** when its complement relative to E, denoted by A^C, is closed in $(E, ^{-})$.

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- Let $(E, ^-)$ be a Tarski space. The set A is **closed** in $(E, ^-)$ when $\overline{A} = A$, and A is **open** when its complement relative to E, denoted by A^C, is closed in $(E, ^-)$.
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- In $(E, ^-)$ every intersection of closed sets is also a closed set.

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- Since, for all $A \subseteq E$, we have $\overline{\overline{A}} = \overline{A}$, then $\overline{A}$ is closed in $(E, ^{-})$.
- In $(E, ^{-})$ every intersection of closed sets is also a closed set.
- $\overline{\emptyset}$ and E correspond to the least and the greatest closed sets, respectively, associated to the consequence operator $^{-}$.

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- The interior $\mathring{A} = \overline{A^C}^C$ is open in $(E, ^-)$.

Tarski spaces

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- If (E, Ω) is an almost topological space and $^{-}$ is a function that maps to the set A its closure $\overline{A}$, then $(E, ^{-})$ is a Tarski space.
- If $(E, ^{-})$ is a Tarski space and $\Omega = \{X^C \subseteq E : X = \overline{X}\}$, then (E, Ω) is an almost topological space.

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- A **TK-algebra** is a sextuple $\mathcal{A} = (A, 0, 1, \vee, \sim, \bullet)$ such that $(A, 0, 1, \vee, \sim)$ is a Boolean algebra and $\bullet$ is a new operator, called **Tarski operator**, such that:
  (i) $a \vee \bullet a = \bullet a$
  (ii) $\bullet a \vee \bullet(a \vee b) = \bullet(a \vee b)$
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- (i) $\Leftrightarrow a \leq \bullet a$
  (ii) $\Leftrightarrow a \leq b \Rightarrow \bullet a \leq \bullet b$.

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Language, axioms and rules

The logic of deductibility - **TK**

Language, axioms and rules

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- (CPC) φ, if φ is a tautology
  (TK_1) $\varphi \rightarrow \blacklozenge\varphi$
  (TK_2) $\blacklozenge\blacklozenge\varphi \rightarrow \blacklozenge\varphi$
  (MP) $\dfrac{\varphi \rightarrow \psi, \varphi}{\psi}$
  ($RM^{\blacklozenge}$) $\dfrac{\vdash \varphi \rightarrow \psi}{\vdash \blacklozenge\varphi \rightarrow \blacklozenge\psi}$.

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- The TK-algebras are algebraic models for the logic **TK**.

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- The rule $RM^{\boxminus}$:

  (RM$^{\boxminus}$) $\dfrac{\vdash \varphi \to \psi}{\vdash \boxminus\varphi \to \boxminus\psi}$.

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Tarski spaces as models for **TK**

Valuation

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- Let $(E, ^{-})$ be a Tarski space. A **restrict valuation** is a function $\langle . \rangle : Var(\mathbf{TK}) \to \mathcal{P}(E)$ that interprets each variable of **TK** in an element of $\mathcal{P}(E)$.

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- A **valuation** is a function $[.] : For(\mathbf{TK}) \to \mathcal{P}(E)$ that extends natural and uniquely $\langle . \rangle$ as follows:
  (i) $[p] = \langle p \rangle$
  (ii) $[\neg\varphi] = E - [\varphi]$
  (iii) $[\blacklozenge\varphi] = \overline{[\varphi]}$
  (iv) $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$
  (v) $[\varphi \vee \psi] = [\varphi] \cup [\psi]$.

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  (iv) $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$
  (v) $[\varphi \vee \psi] = [\varphi] \cup [\psi]$.
- So:
  (vi) $[\top] = E$, where $\top$ is any tautology
  (vii) $[\bot] = \emptyset$, where $\bot$ is any contradiction.

Tarski spaces as models for **TK**

Tarski models

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- Let $(E, ^{-})$ be a Tarski space. A **model** for a set $\Gamma \subseteq For(\mathbf{TK})$ is a valuation $[.] : For(\mathbf{TK}) \to \mathcal{P}(E)$, such that $[\gamma] = E$, for each formula $\gamma \in \Gamma$.

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- In particular, if $\varphi \in For(\mathbf{TK})$, then a valuation $[.] : For(\mathbf{TK}) \to \mathcal{P}(E)$ is a model for φ when $[\varphi] = E$. In this case we write $\langle (E, ^{-}), [.] \rangle \vDash \varphi$.

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- A formula φ is **valid**, what is denoted by $\vDash \varphi$, if for every space $(E, ^{-})$ and every valuation $[.] : For(\mathbf{TK}) \to \mathcal{P}(E)$, we have that $\langle (E, ^{-}), [.] \rangle \vDash \varphi$.

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- If $\Gamma \cup \{\psi\} \subseteq For(\mathbf{TK})$, then Γ **logically implies** ψ, what is denoted by $\Gamma \vDash \psi$, if every model of Γ is a model of ψ.

Tarski spaces as models for **TK**

Adequacy

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- $[\varphi \to \psi] = E \Leftrightarrow [\varphi] \subseteq [\psi]$.

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- $[\varphi \to \psi] = E \Leftrightarrow [\varphi] \subseteq [\psi]$.
- (Adequacy) If $\Gamma \cup \{\varphi\} \subseteq For(\mathbf{TK})$, then $\Gamma \vdash \varphi \Leftrightarrow \Gamma \vDash \varphi$.

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TK plus the modal axiom B

The axiom B

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- This version of axiom **B** is equivalent to: $\blacklozenge \boxminus \varphi \to \varphi$.
- The models of **TK + B** are almost topological spaces (E, Ω) with a constraint:
- To validate $\varphi \to \boxminus \blacklozenge \varphi$ we need that for any $\varphi \in For(\mathbf{TK})$, $v(\varphi \to \boxminus \blacklozenge \varphi) = E$.

TK plus the modal axiom **B**

Consequences

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- So: $v(\varphi \to \boxminus\blacklozenge\varphi) = E \Leftrightarrow v(\neg\varphi \vee \boxminus\blacklozenge\varphi) = E \Leftrightarrow$
  $v(\neg\varphi) \cup v(\boxminus\blacklozenge\varphi) = E \;\Leftrightarrow v(\varphi)^c \cup v(\mathring{\blacklozenge}\varphi) = E \;\Leftrightarrow\; v(\varphi)^c \cup \overline{\mathring{v(\varphi)}} = E.$

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- For each $\varphi \in For(\mathbf{TK})$, $v(\varphi) \subseteq \overline{\mathring{v(\varphi)}}$.
- By the equivalence $\varphi \to \boxminus\blacklozenge\varphi \;\Leftrightarrow\; \blacklozenge\boxminus\varphi \to \varphi$:

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- For each $\varphi \in For(\mathbf{TK})$, $v(\varphi) \subseteq \overline{\mathring{v(\varphi)}}$.
- By the equivalence $\varphi \to \boxminus\blacklozenge\varphi \ \Leftrightarrow \ \blacklozenge\boxminus\varphi \to \varphi$:
- $v(\blacklozenge\boxminus\varphi \to \varphi) = v(\neg\blacklozenge\boxminus\varphi \vee \varphi) = v(\neg\blacklozenge\boxminus\varphi) \cup v(\varphi)$
  $= \overline{\mathring{v(\varphi)}}^{\,c} \cup v(\varphi) = E.$

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- By the equivalence $\varphi \to \boxminus\blacklozenge\varphi \;\Leftrightarrow\; \blacklozenge\boxminus\varphi \to \varphi$:
- $v(\blacklozenge\boxminus\varphi \to \varphi) = v(\neg\blacklozenge\boxminus\varphi \vee \varphi) = v(\neg\blacklozenge\boxminus\varphi) \cup v(\varphi)$
  $= \overline{\mathring{v(\varphi)}}^{c} \cup v(\varphi) = E.$
- Thus, $v(\varphi)^c \subseteq \overline{\mathring{v(\varphi)}}^{c}$.

TK plus the modal axiom **B**

Consequences

- So: $v(\varphi \to \boxminus\blacklozenge\varphi) = E \Leftrightarrow v(\neg\varphi \vee \boxminus\blacklozenge\varphi) = E \Leftrightarrow$
  $v(\neg\varphi) \cup v(\boxminus\blacklozenge\varphi) = E \;\Leftrightarrow v(\varphi)^c \cup v(\mathring{\blacklozenge}\varphi) = E \;\Leftrightarrow\; v(\varphi)^c \cup \overline{v(\mathring{\varphi})} = E.$
- For each $\varphi \in For(\mathbf{TK})$, $v(\varphi) \subseteq \overline{v(\mathring{\varphi})}$.
- By the equivalence $\varphi \to \boxminus\blacklozenge\varphi \;\Leftrightarrow\; \blacklozenge\boxminus\varphi \to \varphi$:
- $v(\blacklozenge\boxminus\varphi \to \varphi) = v(\neg\blacklozenge\boxminus\varphi \vee \varphi) = v(\neg\blacklozenge\boxminus\varphi) \cup v(\varphi)$
  $= \overline{v(\mathring{\varphi})}^c \cup v(\varphi) = E.$
- Thus, $v(\varphi)^c \subseteq \overline{v(\mathring{\varphi})}^c$.
- For every $\varphi \in For(\mathbf{TK})$, we have $\overline{v(\mathring{\varphi})} \subseteq v(\varphi)$.

TK plus the modal axiom **B**

Constraint

TK plus the modal axiom **B**

Constraint

- An almost topological space (E, Ω) is a model for **TK** + **B**, if for every $A \subseteq E$ it holds:
  (i) $A \subseteq \mathring{\overline{A}}$;
  (ii) $\overline{\mathring{A}} \subseteq A$.

TK plus the modal axiom B

Constraint

- An almost topological space (E, Ω) is a model for **TK + B**, if for every $A \subseteq E$ it holds:
  (i) $A \subseteq \overline{\mathring{A}}$;
  (ii) $\mathring{\overline{A}} \subseteq A$.
- If (E, Ω) is a model for **TK + B**, and we consider $f, g : (\mathcal{P}(E), \subseteq) \to (\mathcal{P}(E), \subseteq)$ defined by $f(A) = \overline{A}$ and $g(A) = \mathring{A}$, then the pair $[f, g]$ determines an adjunction (a Galois pair).

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TK plus the modal axiom **5**

The axiom **5**

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The axiom **5**

- The modal axiom **5**: $\blacklozenge\psi \to \boxminus\blacklozenge\psi$.

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- The modal axiom **5**: $\blacklozenge\psi \to \boxminus\blacklozenge\psi$.
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- 1. $\blacklozenge\psi \to \boxminus\blacklozenge\psi$
  2. $\blacklozenge\neg\psi \to \boxminus\blacklozenge\neg\psi$
  3. $\neg\boxminus\blacklozenge\neg\psi \to \neg\blacklozenge\neg\psi$
  4. $\neg\neg\blacklozenge\neg\neg\boxminus\neg\neg\psi \to \boxminus\psi$
  5. $\blacklozenge\boxminus\psi \to \boxminus\psi$. **(5')**

TK plus the modal axiom **5**

The axiom **5**

- The modal axiom **5**: $\blacklozenge\psi \to \boxminus\blacklozenge\psi$.
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  3. $\neg \boxminus \blacklozenge\neg\psi \to \neg\blacklozenge\neg\psi$
  4. $\neg\neg\blacklozenge\neg\neg \boxminus \neg\neg\psi \to \boxminus\psi$
  5. $\blacklozenge \boxminus \psi \to \boxminus\psi$. **(5')**
- **5** $\Rightarrow$ **B**, because $\psi \to \blacklozenge\psi$ and $\boxminus\psi \to \psi$.

TK plus the modal axiom **5**

Consequences over the spaces

TK plus the modal axiom **5**

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- We must to interpret the axiom **5** in an almost topological space by:

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TK plus the modal axiom **5**

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- In an almost topological space (E, Ω) such that for every $A \subseteq E$, we have $\overline{A} = \mathring{\overline{A}}$, then every closed set is also open.

TK plus the modal axiom **5**

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- In an almost topological space (E, Ω) such that for every $A \subseteq E$, we have $\overline{A} = \mathring{\overline{A}}$, then every open set is closed.

TK plus the modal axiom **5**

Space of clopens

Space of clopens

- These conditions say that every closed set is open and every open set is closed, but it say not that every subset of E is open and closed. It can be some set non open and non closed, as in the case that the only open and closed are $\emptyset$ and E.

TK plus the modal axiom **5**

Space of clopens

- These conditions say that every closed set is open and every open set is closed, but it say not that every subset of E is open and closed. It can be some set non open and non closed, as in the case that the only open and closed are $\emptyset$ and E.
- The almost topological spaces such that for any $A \subseteq E$, $\overline{A} = \mathring{\overline{A}}$, are adequate models for **TK** + **5**.

TK plus the modal axiom **5**

Space of clopens

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- The almost topological spaces such that for any $A \subseteq E$, $\overline{A} = \mathring{\overline{A}}$, are adequate models for **TK + 5**.
- If (E, Ω) is an almost topological space in which for every $A \subseteq E$, we have $\overline{A} = \mathring{\overline{A}}$, then (E, Ω) is a topological space.

TK plus the modal axiom **5**

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- The almost topological spaces such that for any $A \subseteq E$, $\overline{A} = \mathring{\overline{A}}$, are adequate models for **TK + 5**.
- If (E, Ω) is an almost topological space in which for every $A \subseteq E$, we have $\overline{A} = \mathring{\overline{A}}$, then (E, Ω) is a topological space.
- The literature on modal logics points that the modal system S_5 has like adequate models exactly the class of topological spaces in which every open set is closed (Kremer, 2009), then S_5 and **TK + 5** has the same valid formulas and, hence, they are deductively coincident.

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