PI theory for Leibniz algebras

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Joint work with

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Leibniz algebras are non-antisymmetric generalization of Lie algebras introduced by Bloh in 1965 and rediscovered by Loday in 1993.

Definition

A Leibniz algebra is a vector space L over a field $\mathbb{K}$ with bilinear product $(-,-)$ which satisfies the Leibniz identity

$$(x,(y,z)) = ((x,y),z) - ((x,z),y). \tag{1}$$

Any Lie algebra is a Leibniz algebra. A Leibniz algebra satisfying the condition $(a,a) = a^2 = 0$ for all $a \in L$ is a Lie algebra and the Leibniz identity becomes the Jacobi identity.

Definition

The Leibniz algebra is called free Leibniz algebra with a set of free generators X and denoted by $Leib(X)$ if, for any Leibniz algebra L, an arbitrary map $X \to L$ can be extended to an algebra homomorphism $Leib(X) \to L$.

Definition

A Leibniz algebra L is said to be nilpotent, if there exists $n \in \mathbb{N}$ such that $L^n = 0$. The minimal number n is said to be the index of nilpotency of L.

Definition

An n-dimensional Leibniz algebra L is called null-filiform if

$$dimL^i = n + 1 - i, 1 \leq i \leq n + 1.$$

Obviously null-filiform Leibniz algebras have maximal index of nilpotency.

An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra

$$NF_n : (e_i, e_1) = e_{i+1},\ 1 \leq i \leq n - 1,$$

where $\{e_1, \ldots, e_n\}$ is a basis for L. (Ayupov and Omirov, 2001)

Remark

Every Leibniz polynomial can be written as a linear combination of left-normed monomials.

PI Algebra

We fix a field K of characteristic zero.

Definition

Let A be an algebra. A *polynomial identity (PI)* for A is a non-zero polynomial $f(x_1, \ldots, x_n)$ in a finite number of non-commuting variables $X_1, \ldots, X_n$ with coefficients in K such that

$$f(a_1, \ldots, a_n) = 0,$$

for all $a_1, \ldots, a_n \in A$.

We shall only consider homogeneous PIs (Recall that a polynomial is homogeneous if it is a linear combination of monomials all having the same degree in each variable).

Example

- An algebra for which there is a PI is called a PI-algebra.
- Any commutative algebra A is a PI-algebra since $f(x, y) = xy - yx$ is PI for A.
- A degree 5 polynomial identity:The polynomial

$$\begin{aligned} f(X, Y, Z) &= [[X, Y]^2, Z] \\ &= XYXYZ - XY^2XZ - YX^2YZ + YXYXZ \\ &- ZXYXY + ZXY^2X + ZYX^2Y - ZYXYX \end{aligned}$$

is a PI for the algebra of 2×2 matrices with entries in K.

T-ideals

Let A be a PI-algebra. The set $Id(A)$ of polynomial identities for A forms a two-sided ideal of the free algebra $K\langle X_1, X_2, \ldots\rangle$.

Definition

A T-ideal is a two-sided ideal of $K\langle X_1, X_2, \ldots\rangle$ that is preserved under all substitution of variables, equivalently, under all algebra endomorphisms of $K\langle X_1, X_2, \ldots\rangle$. The ideal $Id(A)$ of polynomial identities is a T-ideal.

Problem I: Determine the ideal $Id(A)$ for a given PI-algebra A?
This is a difficult problem, solved only for a handful of algebras such as commutative algebras.
Problem II: Is $Id(A)$ generated by a finite number of polynomial identitiesas a T-ideal?
In 1987 Kemer gave a positive answer to the Specht problem for any PI-algebra (whether it is finitely generated or not).

We now turn to a class of algebras with extra structure. In fact, through endowing algebras group grading structures their properties can sometimes be described based on the properties of the structure-induced subspaces of them.

Definition

A Leibniz algebra L is a graded algebra, by means of the abelian group G, if L decomposes as the direct sum of vector subspaces $L = \bigoplus_{g \in G} L_g$.

Example

(Omirov, 2006) Consider the complex (non-Lie) Leibniz algebra L with the basis $\{e, h, f, p, q\}$ defined by the following multiplications

$$
\begin{aligned}
&(e, h) = 2e,\ (h, f) = 2f,\ (e, f) = h,\\
&(h, e) = -2e,\ (f, h) = -2f,\ (f, e) = -h\\
&(p, h) = p,\ (p, f) = q,\\
&(q, h) = -q,\ (q, e) = -p,
\end{aligned}
$$

where omitted products are zero.

The Leibniz algebra L can be $\mathbb{Z}$-graded as $L = \oplus_{z\in\mathbb{Z}} L_z$ such that $L_0 = \langle h \rangle$, $L_1 = \langle p \rangle$, $L_{-1} = \langle q \rangle$, $L_2 = \langle e \rangle$, $L_{-2} = \langle f \rangle$ and $L_z = 0$ for any $z \notin \{0, \pm 1, \pm 2\}$.
Let $L = NF_n$ be a null-filiform Leibniz algebra. There exists the following theorem regarding gradings of NF_n.

Lemma

(Calderon et al. 2019) Let L be a null-filiform Leibniz algebra of dimension n. Then, up to equivalence, all cyclic toral gradings are the following:

(1) *The trivial grading gives by* $L = \langle e_1, \dots, e_n \rangle$;

(2) *The* $\mathbb{Z}$*-grading gives by* $L = \langle e_1 \rangle_1 \oplus \langle e_2 \rangle_2 \oplus \cdots \oplus \langle e_n \rangle_n$;

(3) *For any* $2 \leq i \leq n-1$, *the* $\mathbb{Z}_i$*-grading given by*
$L = L_{\bar{0}} \oplus L_{\bar{1}} \oplus \cdots \oplus L_{\overline{i-1}}$.

The homogeneous subspaces in the previous lemma are described in following sense,

$$
\begin{aligned}
L_{\bar{0}} &= \langle e_i, e_{2i}, \ldots, e_{mi} \rangle \\
L_{\bar{1}} &= \langle e_1, e_{i+1}, \ldots, e_{mi+1} \rangle \\
&\ldots \\
L_{\bar{p}} &= \langle e_p, e_{i+p}, \ldots, e_{mi+p} \rangle \\
L_{\overline{i-1}} &= \langle e_{i-1}, e_{2i-1}, \ldots, e_{(m-1)i+i-1} \rangle .
\end{aligned}
$$

being $n = mi + p$ with $0 \leq p \leq i-1$ and $p \in \mathbb{N}$.

Let consider the case of $\mathbb{Z}_2$-grading of null-filiform Leibniz algebra NF_n with X as the set of free generators. Assume that $X = Y \cup Z$ is a disjoint union of the sets $Y = \{y_1, y_2 \dots\}$ and $Z = \{z_1, z_2, \dots\}$. According to the main lemma about gradings, we have the $\mathbb{Z}_2$-gradings as

$$L := \langle e_2, e_4, \dots, e_n \rangle_0 \oplus \langle e_1, e_3, \dots, e_{n-1} \rangle_1,$$

or

$$L := \langle e_2, e_4, \dots, e_{n-1} \rangle_0 \oplus \langle e_1, e_3, \dots, e_n \rangle_1,$$

where NF_n is of dimension even or odd, respectively.

The $\mathbb{Z}_2$-graded identities

The following polynomials are graded identities for $\mathbb{Z}_2$-graded null-filiform Leibniz algebra NF_n:

1. $y_1 y_2$,
2. zy,
3. $x_1(x_2 x_3)$,

where x_1 and x_2 are any variables and the fourth polynomial identity is only valid for NF_3.

The $\mathbb{Z}_i$-graded identities

We consider the following graded identities for $\mathbb{Z}_i$-gradings. The following polynomials are $\mathbb{Z}_i$-graded identities for NF_n.

1. $mn,\ n \notin L_{\bar{1}}$
2. $w_3(w_1 w_2),\ w_i \in L_{\bar{1}}$

The $\mathbb{Z}$-graded identities

Here we describe generators of the ideal of graded identities for $\mathbb{Z}$-graded null-filiform Leibniz algebras. Let $X^i = \{x_1^i, x_2^i, \dots\}$ $(i \in \mathbb{Z})$ be countable infinite disjoint sets, and put $X = \bigcup_{i\in\mathbb{Z}} X^i$. The elements of X^i are of degree i. Let consider Leibniz polynomials m and n which are of degree i and j and so (m, n) is of degree $i + j$. A polynomial $f(x_{j_1}^{i_1}, \dots, x_{j_k}^{i_k})$ is a graded identity of the $\mathbb{Z}$-graded null-filiform Leibniz algebra $NF_n = \bigoplus_{r\in\mathbb{Z}} L_r$ if $f(a_1, \dots, a_k) = 0$ in L for every choice $a_t \in L_{i_t}$. The following polynomials are graded identities for $\mathbb{Z}$-graded NF_n.

1. $x^i, \quad i \notin \{1, \dots, n\}$
2. $x_1^i x_2^j, \qquad 1 \le i \le n, \quad 2 \le j \le n$
3. $((x_{i_1}^{a_1} x_{i_2}) x_{i_3}) - ((x_{i_1}^{a_1} x_{i_3}) x_{i_2})$ for $a_1 \ge 1$

Question

Do the preceding identities of NF_n generate the ideal of graded identities of null-filiform Leibniz algebras NF_n?

Thank you for your attention.