Ângulo de Grassmann, Produtos de Multivetores, e Teoremas de Pitágoras Generalizados

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Principal Angles between Subspaces

X = real or complex n-dimensional vector space, with inner product $\langle\cdot,\cdot\rangle$.

Definition

Let $V, W \subset X$ be subspaces, $p = \dim V$, $q = \dim W$, $m = \min\{p, q\}$. A singular value decomposition gives orthonormal *principal bases*

$$(e_1, \dots, e_p) \text{ of } V, \qquad (f_1, \dots, f_q) \text{ of } W,$$

in which the orthogonal projection $P : V \to W$ is given by a $q \times p$ diagonal matrix, with the diagonal formed by the $\cos\theta_i$'s of their *principal angles*

$$0 \leq \theta_1 \leq \ldots \leq \theta_m \leq \frac{\pi}{2}.$$

Proposition

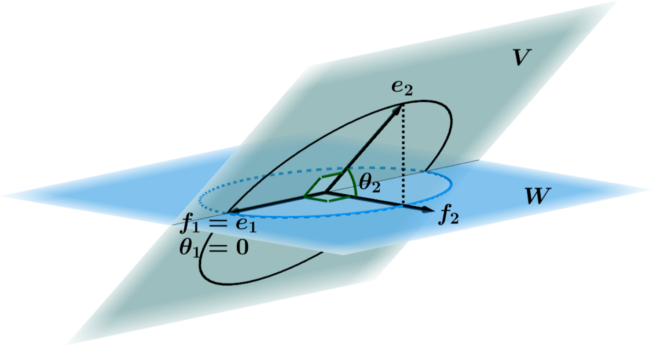
Orthonormal bases $(e_1, \dots, e_p)$ *of* V *and* $(f_1, \dots, f_q)$ *of* W*, and angles* $0 \leq \theta_1 \leq \ldots \leq \theta_m \leq \frac{\pi}{2}$*, constitute principal bases and angles if*

$$\langle e_i, f_j\rangle = \delta_{ij}\cos\theta_i.$$

Geometric interpretation:

- unit sphere of V projects to an ellipsoid in W;
- the e_i's project onto its semi-axes;
- the f_i's point along the semi-axes;
- the semi-axes have lengths $\cos\theta_i$.

In the complex case, for each i there are 2 semi-axes of equal lengths, corresponding to projections of e_i and $\mathrm{i}e_i$.



Example (A)

In $\mathbb{R}^4$,

$$e_1 = (1,0,1,0)/\sqrt{2} \qquad f_1 = (1,0,0,0)$$
$$e_2 = (0,1,0,1)/\sqrt{2} \qquad f_2 = (0,1,0,0)$$

are principal vectors for $V = \text{span}(e_1, e_2)$ and $W = \text{span}(f_1, f_2)$, with principal angles $\theta_1 = \theta_2 = 45°$.

Example (B)

In $\mathbb{C}^4$,

$$e_1 = (\mathrm{i}, \sqrt{3}, 0, 0)/2 \qquad f_1 = (\mathrm{i}, 0, 0, 0)$$
$$e_2 = (0,0,1,1)/\sqrt{2} \qquad f_2 = (0,0,1+\mathrm{i}, 1-\mathrm{i})/2$$

are principal vectors for $V = \text{span}_{\mathbb{C}}(e_1, e_2)$ and $W = \text{span}_{\mathbb{C}}(f_1, f_2)$, with principal angles $\theta_1 = 60°$ and $\theta_2 = 45°$.

Proposition

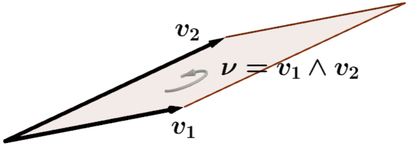
Given pairs of subspaces (V, W) *and* (V', W'), *with* $\dim V' = \dim V$ *and* $\dim W' = \dim W$, *there is an orthogonal/unitary transformation taking* V *to* V' *and* W *to* W' $\Leftrightarrow$ *both pairs have the same principal angles.*

We need all principal angles to describe the relative position of subspaces. But it is often convenient if we can combine them into a single number describing the property of interest.

Grassmann angle

Grassmann algebra

A *(p-)blade* is a decomposable multivector $\nu = v_1 \wedge \ldots \wedge v_p \in \Lambda^p X$.



If $\nu \neq 0$ the vectors are L.I., so ν *determines* the p-dimensional subspace $V = \text{span}(v_1, \ldots, v_p)$, and $\Lambda^p V = \text{span}(\nu)$ is a line in $\Lambda^p X$.

Inner product of p-blades $\nu = v_1 \wedge \ldots \wedge v_p$ and $\omega = w_1 \wedge \ldots \wedge w_p$ is

$$\langle \nu, \omega \rangle = \det \langle v_i, w_j \rangle = \begin{vmatrix} \langle v_1, w_1 \rangle & \cdots & \langle v_1, w_p \rangle \\ \vdots & \ddots & \vdots \\ \langle v_p, w_1 \rangle & \cdots & \langle v_p, w_p \rangle \end{vmatrix}.$$

Real case: $\|\nu\| = \sqrt{\langle \nu, \nu \rangle} =$ p-dim volume of parallelepiped $v_1 \wedge \ldots \wedge v_p$.
Complex case: $\|\nu\|^2 = 2p$-dim volume of $v_1 \wedge_{\mathbb{R}} \mathrm{i}v_1 \wedge_{\mathbb{R}} \ldots \wedge_{\mathbb{R}} v_p \wedge_{\mathbb{R}} \mathrm{i}v_p$.

Grassmann angle

Definition

Let $V, W \subset X$ be subspaces, with principal angles $\theta_1, \ldots, \theta_m$, $P = \mathrm{Proj}^V_W$, and $\mathbf{P}$ = matrix representing P in orthonormal bases.

The *Grassmann angle* $\Theta_{V,W} \in [0, \frac{\pi}{2}]$ of V with W can be defined by:

- $\cos^2 \Theta_{V,W} = \det(\bar{\mathbf{P}}^T \mathbf{P})$.

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- $\cos^2 \Theta_{V,W} = \det(\bar{\mathbf{P}}^T \mathbf{P})$.
- $\cos \Theta_{V,W} = \begin{cases} \cos\theta_1 \cdot \ldots \cdot \cos\theta_m & \text{if } \dim V \leq \dim W, \\ 0 & \text{if } \dim V > \dim W. \end{cases}$

Example (A')

In ▸ Example A, all vectors in V make a 45° angle with W, but $\Theta_{V,W} = 60°$.

Definition (cont.)

- Real case: $\cos \Theta_{V,W} = \dfrac{\operatorname{vol} P(S)}{\operatorname{vol} S}$, for any measurable set $S \subset V$.

  Complex: $\cos^2 \Theta_{V,W} = \dfrac{\operatorname{vol} P(S)}{\operatorname{vol} S}$.

Definition (cont.)

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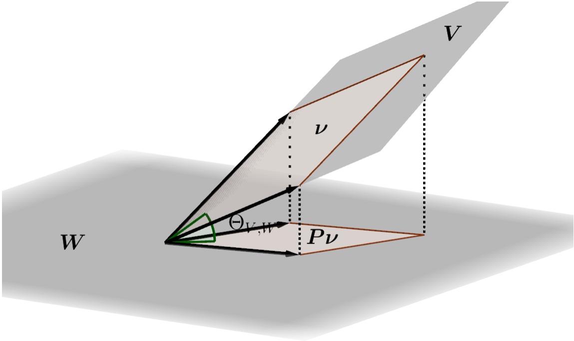
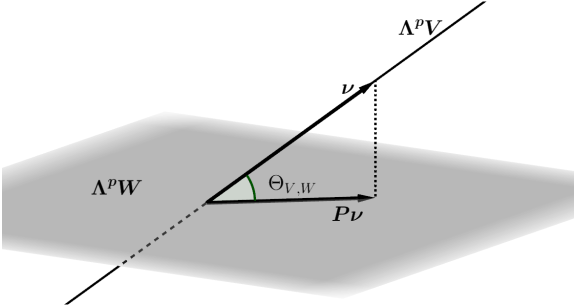
- $\cos \Theta_{V,W} = \dfrac{\|P\nu\|}{\|\nu\|}$, for any $\nu \in \Lambda^p V$, $p = \dim V$.



Definition (cont.)

- $\Theta_{V,W}$ = angle in $\Lambda^p X$ between the line $\Lambda^p V$ and the subspace $\Lambda^p W$.



Proposition

- $\Theta_{V,W} = 0 \;\Leftrightarrow\; V \subset W$.
- $\Theta_{V,W} = \frac{\pi}{2} \;\Leftrightarrow\; \exists v \in V,\ v \neq 0$, *such that* $v \perp W$.

In general this angle is asymmetric, $\Theta_{V,W} \neq \Theta_{W,V}$.

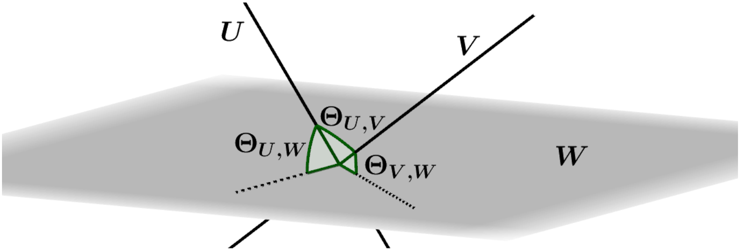
Proposition

If $\dim V = \dim W$ *then* $\Theta_{V,W} = \Theta_{W,V}$.

Theorem (Triangle Inequality)

For any subspaces $U, V, W \subset X$,

$$\Theta_{U,W} \leq \Theta_{U,V} + \Theta_{V,W}.$$



$\Theta_{V,W}$ = Fubini-Study distance in the Grassmannian of p-dim subspaces, and gives quasi-pseudo-metric in the total Grassmannian of all subspaces.

Complementary Grassmann angle

Definition

Complementary Grassmann angle $\Theta^{\perp}_{V,W} \in [0, \frac{\pi}{2}]$ of V and W is the Grassmann angle of V with the orthogonal complement of W,

$$\Theta^{\perp}_{V,W} = \Theta_{V,W^{\perp}}.$$

Reason for special notation: it has symmetry that was absent in $\Theta_{V,W}$.

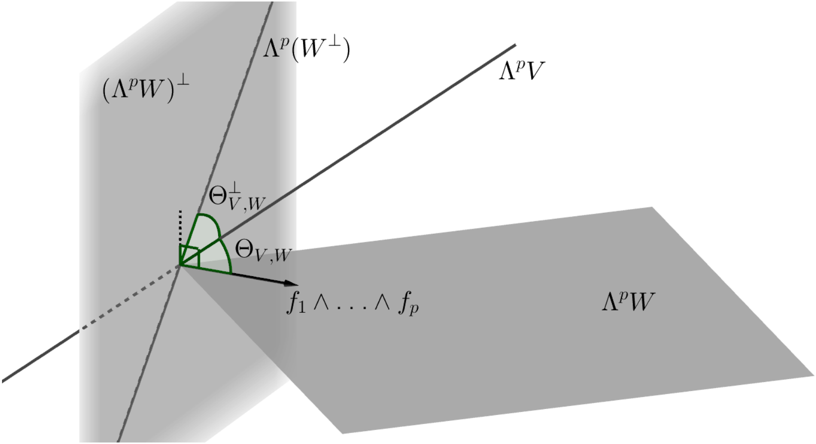
Proposition

$$\Theta^{\perp}_{V,W} = \Theta^{\perp}_{W,V}.$$

Example (A')

In Example A, all vectors in V make a 45° angle with both W and $W^{\perp}$, so $\Theta_{V,W} = \Theta^{\perp}_{V,W} = 60°$.

In general, $\Theta^{\perp}_{V,W} \neq \frac{\pi}{2} - \Theta_{V,W}$, because $\Lambda^p(W^{\perp}) \neq (\Lambda^p W)^{\perp}$.



Proposition

For a line L, the complementary Grassmann angle is the usual complement,

$$\Theta^{\perp}_{L,W} = \frac{\pi}{2} - \Theta_{L,W},$$

so that

$$\cos \Theta^{\perp}_{L,W} = \sin \Theta_{L,W}.$$

But for higher dimensional subspaces the projection on $W^{\perp}$ will be given by a product of sines.

Theorem

If $\theta_1, \ldots, \theta_m$ are the principal angles of V and W then

$$\cos \Theta^{\perp}_{V,W} = \sin \theta_1 \cdot \ldots \cdot \sin \theta_m.$$

Blade Products

Inner and Exterior products of blades

Theorem

For any blades $\nu, \omega \in \Lambda^p X$*, determining* $V, W \subset X$*,*

$$\langle \nu, \omega \rangle = \sigma_{\nu,\omega} \|\nu\| \|\omega\| \cos \Theta_{V,W}.$$

$\sigma_{V,W}$ is a sign ± 1 (real case) or phase factor $e^{i\varphi}$ (complex case).
It appears because we defined $\Theta_{V,W} \in [0, \frac{\pi}{2}]$, for non-oriented subspaces.

Theorem

For any blades $\nu, \omega \in \Lambda X$*, determining* $V, W \subset X$*,*

$$\|\nu \wedge \omega\| = \|\nu\| \|\omega\| \cos \Theta^{\perp}_{V,W}.$$

Interior product of blades

Definition

Interior product $\nu \lrcorner \omega$ of blades $\nu \in \Lambda^p X$ and $\omega \in \Lambda^q X$, with $p \leq q$, is the unique element of $\Lambda^{q-p} X$ such that

$$\langle \nu \lrcorner \omega, \tau \rangle = \langle \omega, \nu \wedge \tau \rangle \text{ for all } \tau \in \Lambda^{q-p} X.$$

Theorem

Given blades $\nu \in \Lambda^p X$ and $\omega \in \Lambda^q X$, with $p \leq q$, determining $V, W \subset X$, and a principal basis $(f_1, \ldots, f_q)$ of W with respect to V,

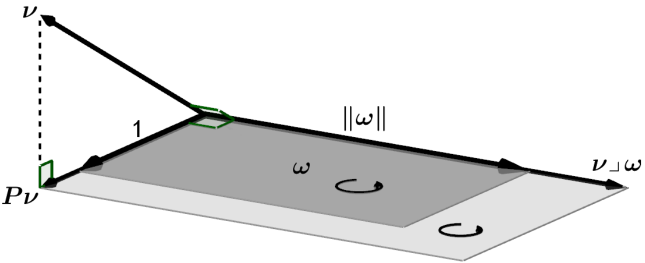
$$\nu \lrcorner \omega = \sigma_{\nu,\omega} \|\nu\| \|\omega\| \cos \Theta_{V,W} \cdot f_{p+1} \wedge \ldots \wedge f_q.$$

$\nu \lrcorner \omega$ is a partial inner product, of ν with a subblade of ω where it projects, leaving another subblade of ω orthogonal to ν.

Theorem

Let $\nu \in \Lambda^p X$, $\omega \in \Lambda^q X$ be blades, with $p \leq q$, determining $V, W \subset X$, and $P = \mathrm{Proj}_W$. Then $\nu \lrcorner \omega$ is characterized by:

- *$\nu \lrcorner \omega$ is a $(q-p)$-subblade of ω completely orthogonal to ν;*
- *$(P\nu) \wedge (\nu \lrcorner \omega)$ has the same orientation of ω;*
- $\dfrac{\|\nu \lrcorner \omega\|}{\|P\nu\|} = \dfrac{\|\omega\|}{1}$.



Trigonometric identities

Theorem

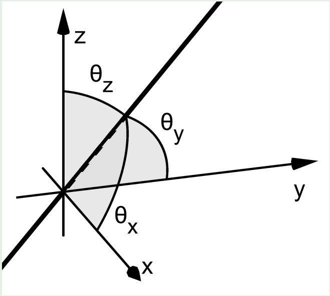
Given a line $L \subset X$ and an orthogonal partition $X = V_1 \oplus \cdots \oplus V_k$,

$$\sum_{j=1}^{k} \cos^2 \Theta_{L,V_j} = 1.$$

Example (Direction cosines)

If $\theta_x, \theta_y, \theta_z$ are the angles of a line in $\mathbb{R}^3$ with the x, y, z axes then

$$\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1.$$



Theorem

For any p-dimensional subspace $V \subset X$,

$$\sum_I \cos^2 \Theta_{V,W_I} = 1,$$

where the W_I's are all p-dimensional coordinate subspaces of an ortonormal basis $\beta = \{w_1, \ldots, w_n\}$ of X, i.e.

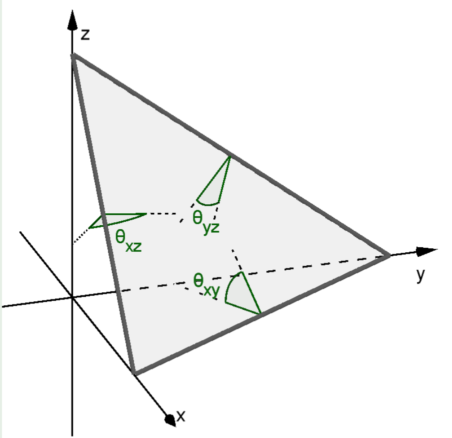
$$W_I = \operatorname{span}(w_{i_1}, \ldots, w_{i_p}),$$

for any multi-index $I = (i_1, \ldots, i_p)$ with $1 \leq i_1 < \ldots < i_p \leq n$.

Example

If $\theta_{xy}, \theta_{xz}, \theta_{yz}$ are the angles of a plane in $\mathbb{R}^3$ with the coordinate planes then

$$\cos^2\theta_{xy} + \cos^2\theta_{xz} + \cos^2\theta_{yz} = 1.$$



Generalized Pythagorean theorems

As Grassmann angles measure volume contraction, these identities give generalized Pythagorean theorems.

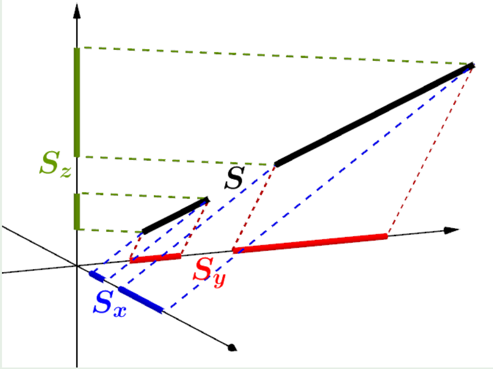
Theorem (Pythagorean theorem for lines)

Given a line $L \subset X$, an orthogonal partition $X = V_1 \oplus \cdots \oplus V_k$, and a measurable set $S \subset L$, with orthogonal projection S_j on V_j,

- *Real case:* $\text{lenght}(S)^2 = \sum \text{lenght}(S_j)^2$;
- *Complex:* $\text{area}(S) = \sum \text{area}(S_j)$.

Complex case: the measure has twice the dimension, but is not squared.

Example

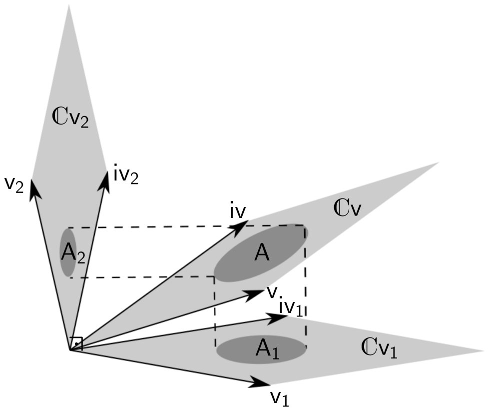


$$\text{lenght}(S)^2 = \text{lenght}(S_x)^2 + \text{lenght}(S_y)^2 + \text{lenght}(S_z)^2$$

Example

Let v_1, v_2 be orthogonal unit complex vectors, and $v = c_1 v_1 + c_2 v_2$ for $c_1, c_2 \in \mathbb{C}$ with $|c_1|^2 + |c_2|^2 = 1$. Any area A in $\mathbb{C}v$ projects to areas $A_1 = |c_1|^2 \cdot A$ in $\mathbb{C}v_1$ and $A_2 = |c_2|^2 \cdot A$ in $\mathbb{C}v_2$, with

$$A = A_1 + A_2.$$



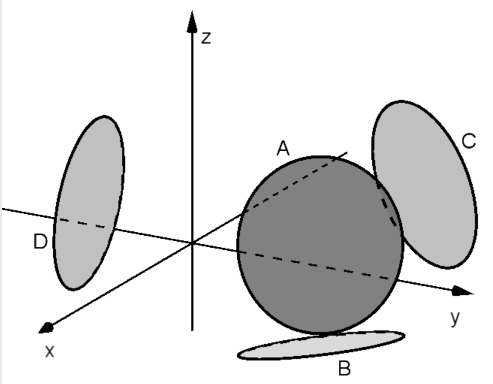
Theorem (Pythagorean theorem for subspaces)

Given a measurable set S in a subspace $V \subset X$, with $\dim V = p$,

- *Real case:* $\mathrm{vol}_p(S)^2 = \sum \mathrm{vol}_p(S_I)^2$,
- *Complex:* $\mathrm{vol}_{2p}(S) = \sum \mathrm{vol}_{2p}(S_I)$,

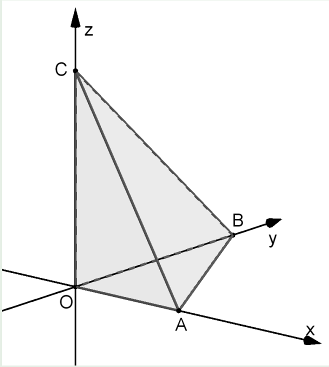
where the S_I's are the orthogonal projections of S on all p-dimensional coordinate subspaces of an orthogonal basis of X.

Example



$$\text{area}(A)^2 = \text{area}(B)^2 + \text{area}(C)^2 + \text{area}(D)^2$$

Example



$$\text{area}(ABC)^2 = \text{area}(OAB)^2 + \text{area}(OAC)^2 + \text{area}(OBC)^2$$

Obrigado

Referências

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