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Reductions between certain incidence problems and the Continuum Hypothesis

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Abstract

In this work, we consider two families of incidence problems, $\mathcal{C}_1$ and $\mathcal{C}_2$, which are related to real numbers and countable subsets of the real line. In both cases, to solve the problem one has to give an appropriate (but random[1]) response to a certain initial data; notice that such procedure could also be interpreted as some one-round game between two players, were the first player gives the initial data and the second player wins if he gives a response which solves the problem. Problems of $\mathcal{C}_1$ are as follows: given a real number x, pick randomly a countably infinite set of reals A hoping that $x \in A$, whereas problems in $\mathcal{C}_2$ are as follows: given a countably infinite set of reals A, pick randomly a real number x hoping that $x \notin A$. One could arguably defend that, at least intuitively, problems of $\mathcal{C}_2$ are easier to solve than problems of $\mathcal{C}_1$. Our main results are the following:

(a) After some suitable formalization, we prove (within **ZFC**) that, on one hand, problems of $\mathcal{C}_2$ are, indeed, at least as easy to solve as problems of $\mathcal{C}_1$.

(b) On the other hand, the statement "Problems of $\mathcal{C}_1$ have the exact same complexity of problems of $\mathcal{C}_2$" is shown to be an equivalent of the Continuum Hypothesis (**CH**).

The suitable formalization for the notion of comparison of complexities between problems will be given by *reductions*: for instance, problems in $\mathcal{C}_2$ will be shown to be *simpler* (or *not more complicated*) than the ones in $\mathcal{C}_1$ because we give a **ZFC** proof that the (act of) solving a problem in $\mathcal{C}_2$ may be reduced to solving a (corresponding) problem in $\mathcal{C}_1$. Those reductions will be given in terms of morphisms between objects of the category $\mathcal{PV}$, which is a subcategory of the dual of the simplest case of the Dialectica Categories introduced by Valeria de Paiva ([4,5]); such morphisms are also known as *Galois-Tukey connections*, which is a terminology due to Peter Vojtáš ([9]). Several connections between this category and Set Theory have been extensively studied by

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[1]This paper deals with some problems which may be viewed as *thought experiments*. In order to state such thought experiments, we have chosen to proceed with a certain *ad hoc* identification, mostly by stylistic reasons: we will identify *randomness* with *arbitrariness*. In practice, such identification consists in the assumption that, when we declare that we are picking arbitrary objects, then there is no pattern involved and all possible outcomes are unpredictable since they obey an equal probability distribution.

Andreas Blass in the 90's (see e.g. [1,2]), and, more recently, have also been investigated by the author ([7,8]). A full description of how the morphisms of $\mathcal{PV}$ correspond to reductions between certain problems may be found at the pages 62 and 63 of [1].

And, as randomly taken countable sets of real numbers may be regarded (in a certain oriented thought experiment) as *the set of punctures of a countable set of darts thrown at the real line*, the proof of our announced equivalence for the Continuum Hypothesis is, in fact, pretty similar to the one presented by Freiling in [3] – whose mathematical content, however, is due to Sierpiński, in his classical monograph on **CH** ([6]). However, in our context Freiling's controversial assumption of symmetry seems unnecessary, and our approach lead us, apparently, to an even more dramatic discussion – if one considers the following question:

- Before being given a countable set A of reals and a real number x, both to be randomly taken, should one say that it will be *easier* (or it will be *more likely*) that, eventually, this real number x *will miss* the countable set A ? Or should one say that, under the very same conditions and interpretations, *it will hit it* ?

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