The Mandelbrot set and its satellite copies

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IMPA

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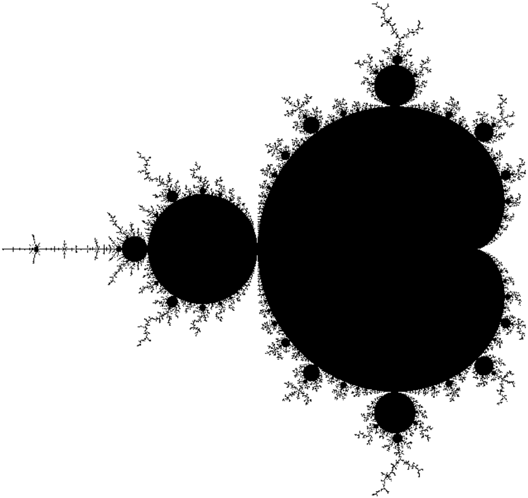
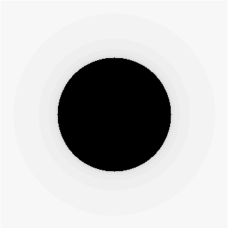
The Mandelbrot set M

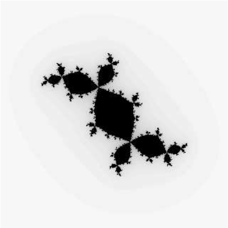
The Mandelbrot set $\mathcal{M}$ is a fractal which classifies the behaviour of complex polynomials



Dynamics of quadratic polynomials on $\widehat{\mathbb{C}}$

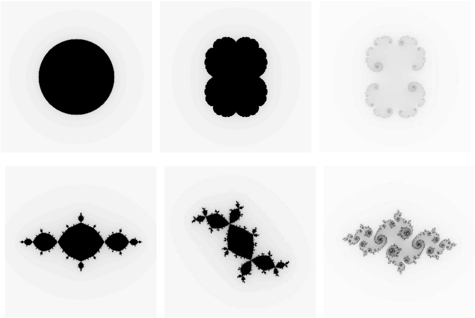
- $P_c(z) = z^2 + c$, ∞ (super)attracting fixed point, with basin $\mathcal{A}_c(\infty)$.
- Filled Julia set $\mathcal{K}_c = \widehat{\mathbb{C}} \setminus \mathcal{A}_c(\infty)$, $J_c = \partial\mathcal{K}_c = \partial\mathcal{A}_c(\infty)$





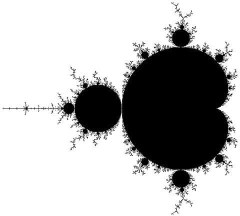
Quadratic polynomials on $\widehat{\mathbb{C}}$

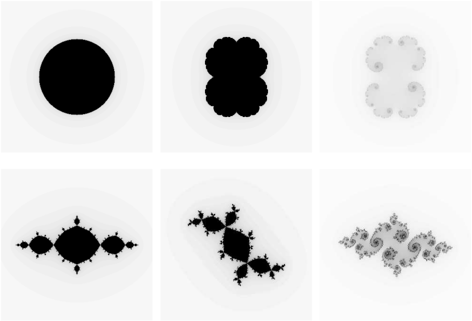
- $P_c(z) = z^2 + c$, ∞ (super)attracting fixed point, with basin $\mathcal{A}_c(\infty)$.
- Filled Julia set $\mathcal{K}_c = \widehat{\mathbb{C}} \setminus \mathcal{A}_c(\infty)$, $J_c = \partial \mathcal{K}_c = \partial \mathcal{A}_c(\infty)$
- The 'Böttcher map' B conjugates P_c to $z \to z^2$ on a nbh of ∞,
- if $\mathcal{K}_c$ is connected, B extends to $B : \mathbb{C} \setminus \mathcal{K}_c \to \mathbb{C} \setminus \overline{\mathbb{D}}$.



The Mandelbrot set $\mathcal{M}$

The Mandelbrot set $\mathcal{M}$ is the set of parameters for which $\mathcal{K}_c$ is connected, (**connectedness locus** for the family P_c).



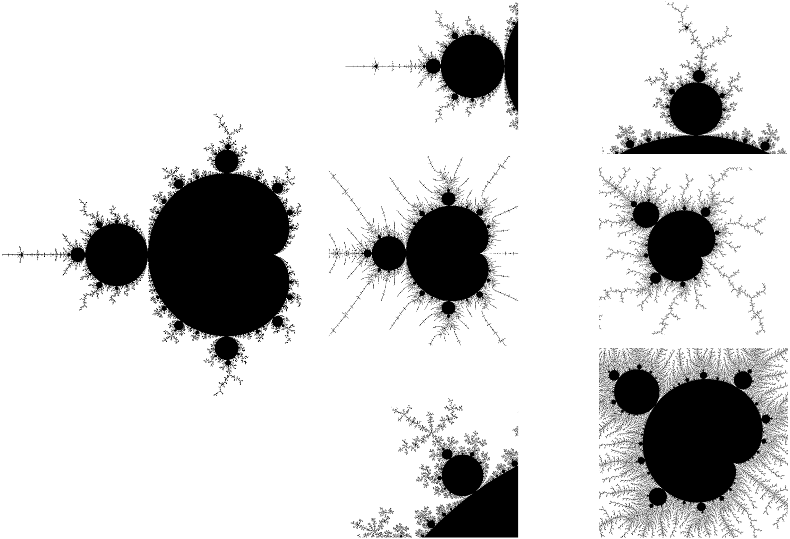


Thm (Fatou) $\mathcal{K}_c$ connected $\Leftrightarrow$ (all finite) critical point(s) $\in \mathcal{K}_c$, so

$$\mathcal{M} = \{c \in \mathbb{C} \mid P_c^n(0) \not\to \infty\}$$

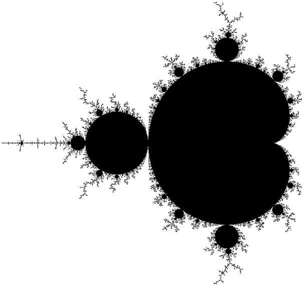
Little copies of $\mathcal{M}$ inside $\mathcal{M}$

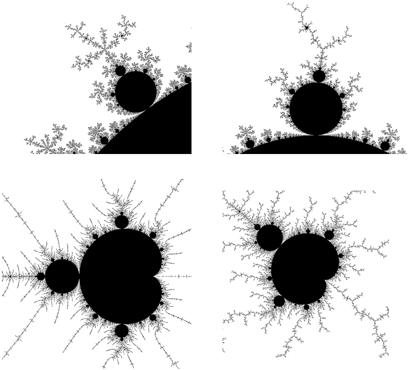
- $\mathcal{M}$ probably best known fractal.
- Striking: little copies of $\mathcal{M}$ in $\mathcal{M}$.



Satellite vs primitive copies

- There are 2 different kind of copies of $\mathcal{M}$ in $\mathcal{M}$:
  - Primitive copies (with a cusp)
  - Satellite copies of $\mathcal{M}$ (without a cusp).



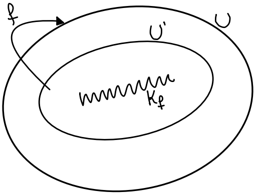


The presence of copies of the Mandelbrot set in the Mandelbrot set is explained by theory of polynomial-like mappings (Douady-Hubbard, '85)

Polynomial-like mappings

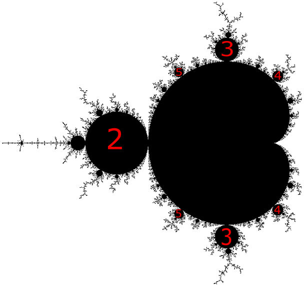
A (deg d) polynomial-like map is a triple (f, U', U), where $U' \subset\subset U$ and $f : U' \to U$ is a (deg d) proper and holomorphic map, and

$$\mathcal{K}_f = \{z \in U' \mid f^n(z) \in U',\ \forall n \geq 0\}$$



- Deg d polynomial restrict to deg d polynomial-like map.

Non trivial examples of polynomial-like mappings

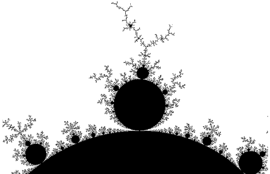


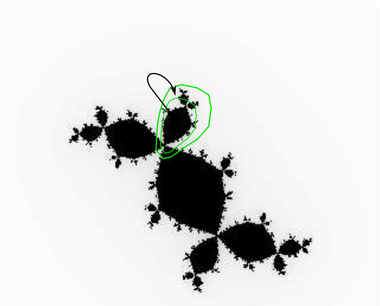
If P_c has an attracting cycle, $c \in \mathring{\mathcal{M}}$

H_q hyperbolic component of $\mathring{\mathcal{M}}$,

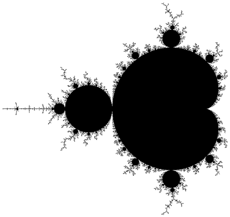
$\forall c \in H_q,\ P_c$ has an attracting cycle of period q.

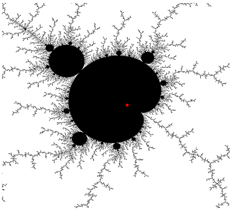
Conjecture: $\mathring{\mathcal{M}} = \bigcup_q H_q$

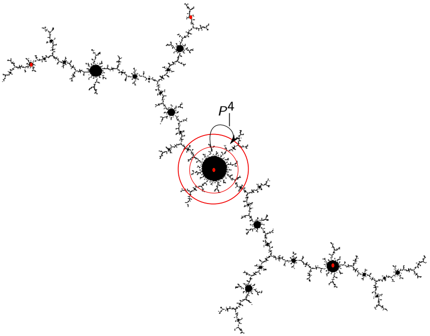




Non trivial examples of polynomial-like mappings



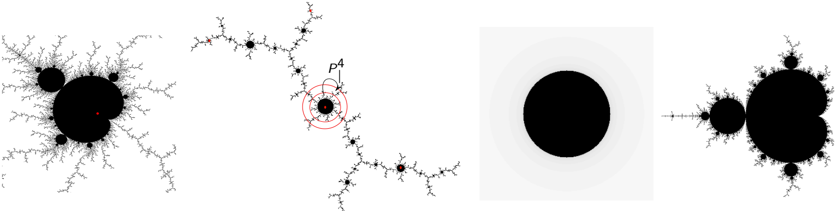




Polynomial-like mappings are polynomials

Straightening theorem (Douady-Hubbard, '85),

Every (dg d) polynomial-like map $f : U' \to U$ is hybrid equivalent (conjugation qc and conformal on $\mathring{\mathcal{K}}_f$) to a (dg d) polynomial, unique if $\mathcal{K}_f$ is connected.



A homeomorphism $\phi : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is K-quasiconformal if:

1. it has distributional partial derivatives $\partial\phi$ and $\overline{\partial}\phi$ which are in L^2_{loc}
2. $|\overline{\partial}\phi| < k|\partial\phi|$ almost everywhere, with $k = (K-1)/(K+1)$.

A conjugacy ϕ is hybrid if it is qc with $\overline{\partial}\phi = 0$ on $\mathcal{K}_f$.

Little copies of the Mandelbrot set

Let $\Lambda \approx \mathbb{D}$, $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of degree 2 polynomial-like map, depending holomorphically on λ, and define $\mathcal{M}_\lambda = \{\lambda \mid \mathcal{K}_\lambda \text{ is connected}\}$.
The straightening theorem induces a map

$$\chi : \mathcal{M}_\lambda \to \mathcal{M},$$

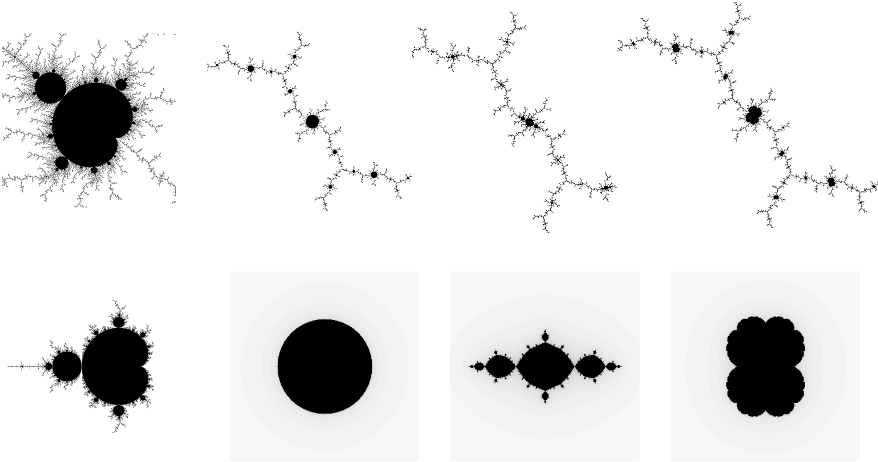
$$\lambda \to c : f_\lambda \sim P_c$$

- **Theorem (D-H,'85)** If $\mathcal{M}_\lambda \subset\subset \Lambda$ (and some other conditions) the map χ is a dynamical homeomorphism.

- **Remark** on hyperbolic components χ is holomorphic.

Primitive copies of $\mathcal{M}$ inside $\mathcal{M}$

- Every P_c in a nbh of a primitive copy has a polynomial-like map restriction, so primitive copies are homeomorphic to the whole $\mathcal{M}$.
- Lyubich, ('99): This homeomorphism is quasiconformal.



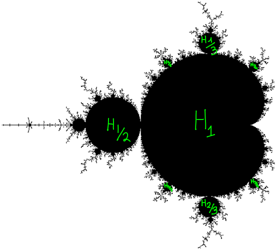
Birth of Satellites copies

$\heartsuit$ (or H_1): P_c has an attracting fixed point (α f.p.),

At $\partial\heartsuit \cap \partial H_{p/q} = root_{p/q}$,
α collides with a p/q-repelling cycle:
$P'_{root_{p/q}}(\alpha) = e^{2\pi i p/q}$.

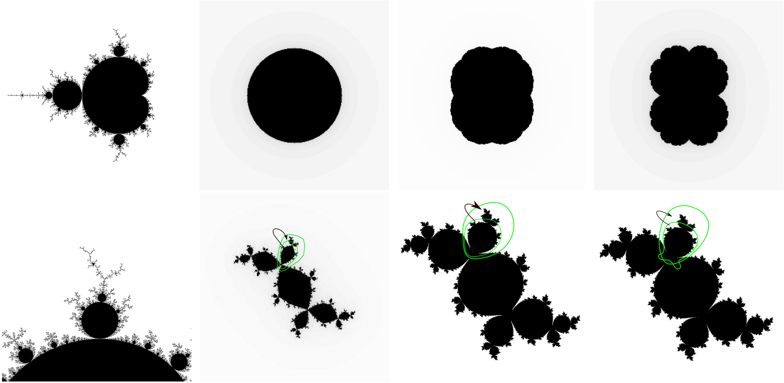
In $H_{p/q}$, the p/q-cycle is attracting
and α repelling.

$\mathcal{M}_{p/q}$ sat. copy attached to $\heartsuit$ at $root_{p/q}$,
so $\heartsuit_{\mathcal{M}_{p/q}} = H_{p/q}$,
p/q
is called the rotation number of the satellite copy.

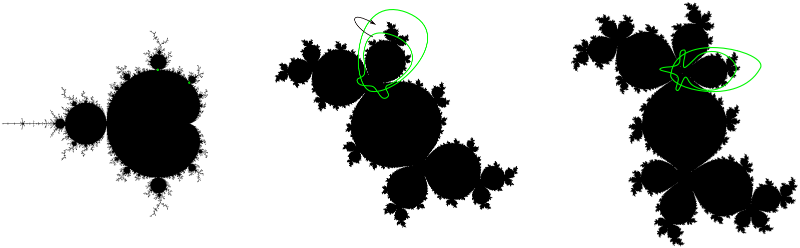


Satellite copies of $\mathcal{M}$ inside $\mathcal{M}$

- Satellite roots are not polynomial-like maps: D-H,('85): Satellite copies are homeomorphic to $\mathcal{M}$ outside any neighbourhood of the root.
- Lyubich, ('99): This homeomorphism is quasiconformal.
- Haissinsky ('00): χ homeomorphism at the root in the satellite case.



Satellite roots



L ('14): roots of any 2 satel. copies have restrictions qc conj.

Are the satellite copies mutually qc homeomorphic?

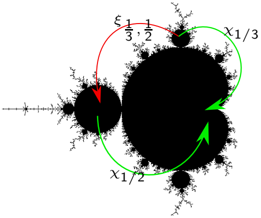
$\mathcal{M}_{p/q}$ satellite copy attached to ♡ at c, where P_c has a fixed point with multiplier $\lambda = e^{2\pi i p/q}$

The homeomorphism

$$\xi_{\frac{p}{q},\frac{P}{Q}} := \chi_{P/Q}^{-1} \circ \chi_{p/q} : \mathcal{M}_{p/q} \to \mathcal{M}_{P/Q}$$

is qc away from nbh of root, and in dynamical plane, the roots are hybrid conjugate on corresponding 'ears'.

Does it extend qc to the root?



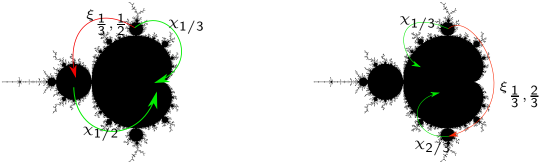
'Presumably the whole satellite set is qc equivalent to the 'one half' of the Mandelbrot set of the family $z \to \lambda z + z^2$'

Are the satellite copies mutually qc homeomorphic?

Theorem (L-Petersen, Invent. Math. '17) For p/q and P/Q irreducible rationals with $q \neq Q$,

$$\xi_{\frac{p}{q},\frac{P}{Q}} := \chi_{P/Q}^{-1} \circ \chi_{p/q} : \mathcal{M}_{p/q} \to \mathcal{M}_{P/Q}$$

is not quasi-conformal.



Theorem (L-Petersen, 2024) For p/q and p'/q irreducible rationals,

$$\xi_{\frac{p}{q},\frac{p'}{q}} := \chi_{p'/q}^{-1} \circ \chi_{p/q} : \mathcal{M}_{p/q} \to \mathcal{M}_{p'/q}$$

is quasi-conformal.

Strategy of the proofs

A homeomorphism $\phi : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is K-quasiconformal if

1. it has distributional partial derivatives $\partial\phi$ and $\overline{\partial}\phi$ which are in L^2_{loc}
2. $|\overline{\partial}\phi| < k|\partial\phi|$ almost everywhere, with $k = (K-1)/(K+1)$.

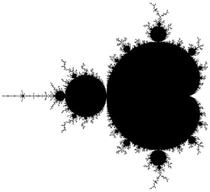
To prove that ξ does extend quasiconformally to the root, we need to prove that K_ξ stays bounded when approaching the root. Strategy:

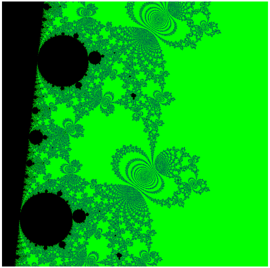
1. Find an upper bound in dynamical plane:
   construct family of uniformly hybrid conjugacies:
   qc conjugacies ϕ_λ with $\overline{\partial}\phi_\lambda = 0$ on $\mathcal{K}_\lambda$, $K_{\phi_\lambda} \leq K$, for all $\lambda \in \mathcal{M}_{p/q} \setminus \{root\}$
2. Translate it to an upper bound in parameter plane:
   $\xi_{\frac{p}{q},\frac{P}{Q}} := \chi^{-1}_{P/Q} \circ \chi_{p/q} : \mathcal{M}_{p/q} \to \mathcal{M}_{P/Q}$ has $K_\xi \leq K$
   (*Plough in the dynamical plane, and harvest in parameter space*)

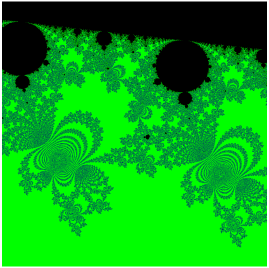
(L-Petersen, '17) Looking for upper bounds, we discovered lower bounds, which, if $Q \neq q$, go to infinity while approaching the root of the satellite copy

Geometric interpretation of the negative result

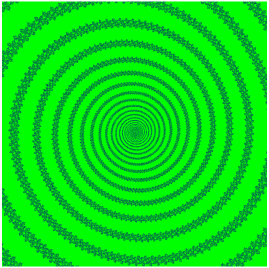
In the right: the Mandelbrot set M.
Center: zoom of M about the 56/57-sublimb $\mathrm{L}^{1/2}_{56/57}$ of the satellite copy $M_{1/2}$ (the one at the bottom).
Left: zoom of M about the 56/57-sublimb $\mathrm{L}^{1/3}_{56/57}$ of the satellite copy $M_{1/3}$ (the one at the left).

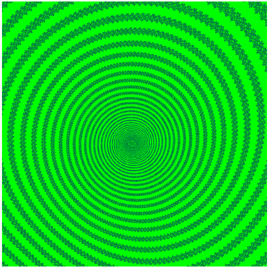






Geometric interpretation

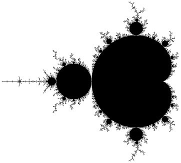




At λ^*, $M_{p/q}$ as. log. spiral $\mathrm{e}^{t\Lambda^*}$, 'step' $\mathrm{e}^{\Lambda^*} = (\lambda^*)^q$,
at ν^*, $M_{P/Q}$ is asympt. e^{tN^*}, 'step' $\mathrm{e}^{N^*} = (\nu^*)^Q$.
$N^* \approx \frac{Q}{q}\Lambda^*$: **these spirals spiral at different speed!**

$\mathcal{M}_{p/q}$ and $\mathcal{M}_{p'/q}$ are qc homeomorphic: proof

Let $\{f_\lambda\}_{\lambda \in \mathcal{M}_{1/3} \setminus \{root\}}$ and $\{g_\nu\}_{\nu \in \mathcal{M}_{2/3} \setminus \{root\}}$

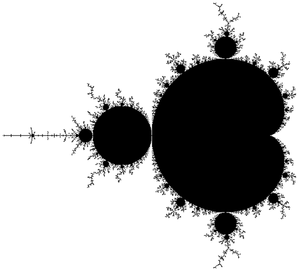


1. Construct a family of uniformly hybrid conjugacies $\phi_\lambda : U_\lambda \to U_\nu$, $\lambda \in \mathcal{M}_{1/3} \setminus \{root\}$, between correspondent polynomial-like maps (this is, qc conjugacies ϕ_λ with $\overline{\partial}\phi_\lambda = 0$ on $\mathcal{K}_\lambda$, $K_{\phi_\lambda} \leq K$, for all $\lambda \in \mathcal{M}_{1/3} \setminus \{root\}$) .

2. Straighten Lyubich space of polynomial-like germs, obtaining that $\xi : \mathcal{M}_{1/3} \to \mathcal{M}_{2/3}$ has $K_\xi \leq K + \epsilon$

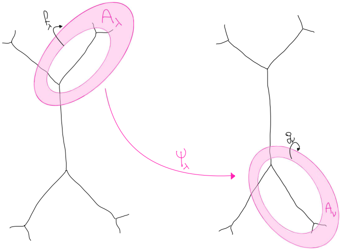
Rk: ξ is holomorphic on hyperbolic components: enough to make construction out of main hyperbolic component!

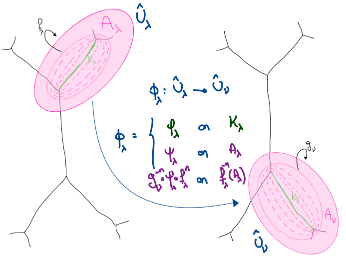
Thank you for your attention!



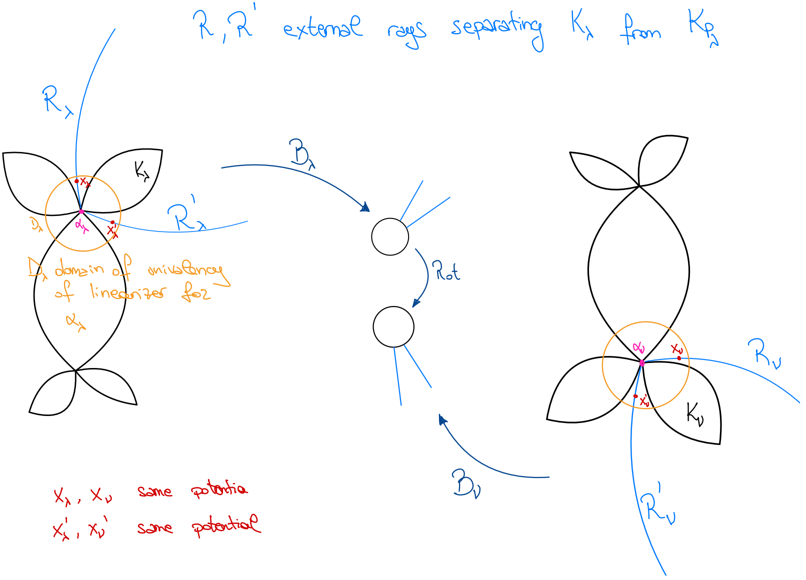
1- Construct a family of uniformly hybrid conjugacies ϕ_λ

- $\{f_\lambda\}_{\lambda \in M_{p/q} \setminus \{root\}}$ and $\{g_\nu\}_{\nu \in M_{p'/q} \setminus \{root\}}$ quadratic-like families, then $\forall\, \lambda\ \exists\ \varphi_\lambda$ hybrid conjugacy (qc and with $\overline{\partial} \varphi_{\lambda | K_\lambda} = 0$) between f_λ and $g_{\nu = \xi(\lambda)}$
- Enough to construct family of uniformly qc external conjugacies: uniformly qc maps $\psi_\lambda : A_\lambda \to A_\nu$ between suitable fundamental annuli A_λ and A_ν.



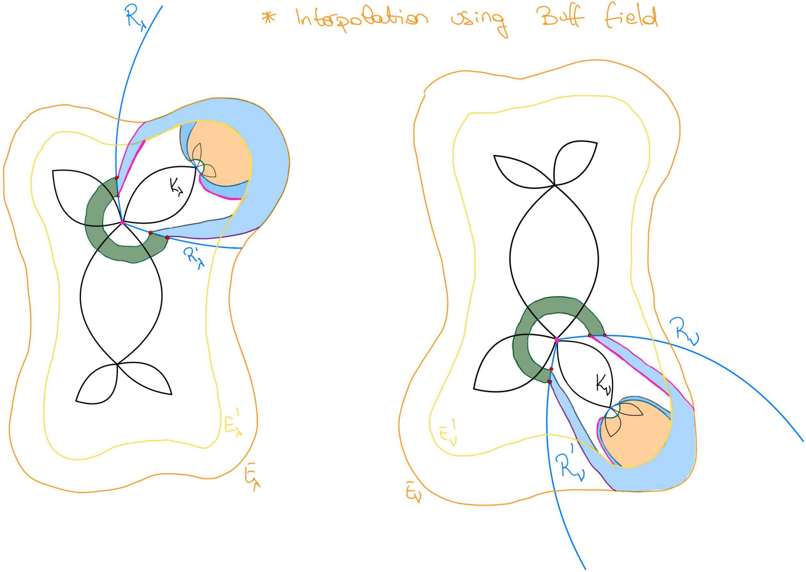


Construction of the fundamental annuli

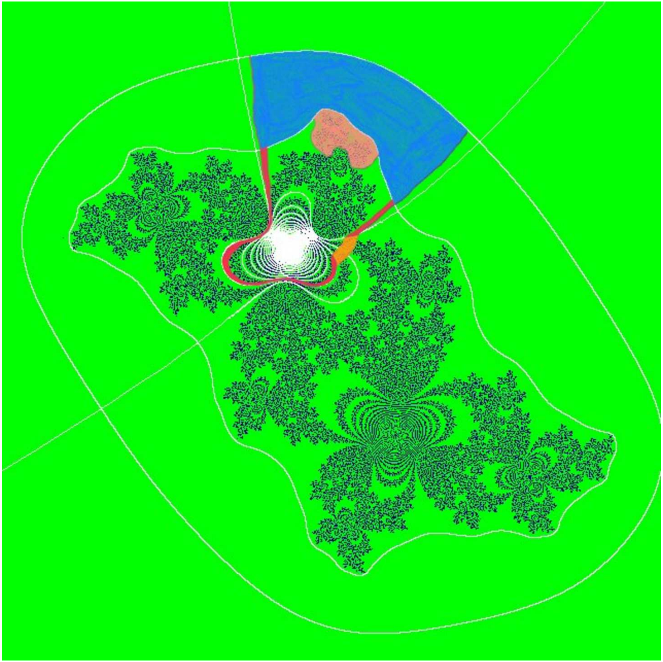


Construction of the fundamental annulus

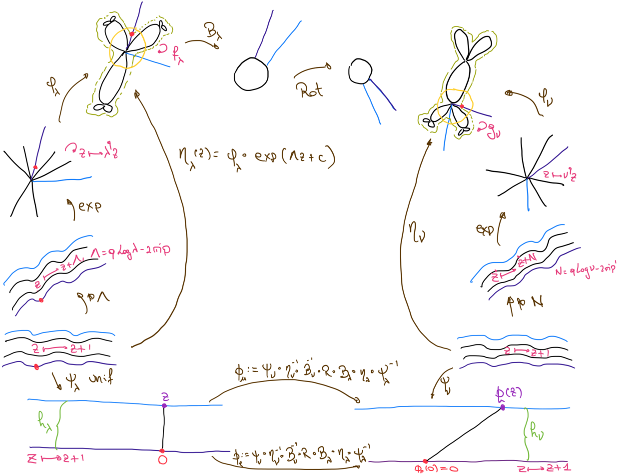
- Interpolation using Böttcher coordinates
- * Interpolation using Buff field



Construction of the fundamental annulus



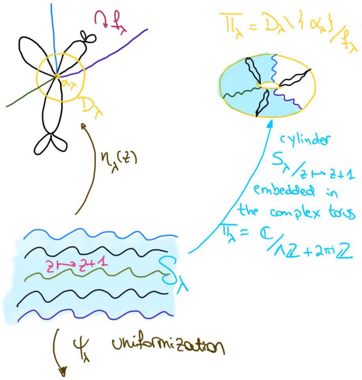
Interpolation Φ_λ between rays



$\Phi_\lambda(x+iy) = \phi_\lambda^\ell(x) + \frac{y}{h_\lambda}(\phi_\lambda^u(x+ih_\lambda) - \phi_\lambda^\ell(x))$. For unif. qc we need:

1. $\exists K_1,\ K_2$ s.t. $\forall \lambda,\ \nu = \xi(\lambda)$, $\quad 0 < K_1 < h_\lambda < K_2, \quad 0 < K_1 < h_\nu < K_2;$
2. $\exists K_3,\ K_4,\ K_5,\ K_6$ s.t. $\quad |\phi_\lambda^u(z) - z| < K_3, \quad K_4 < |(\psi_i)'| < K_5, \quad |\psi_i(z) - z| < K_6$

1- Bounding heights



$h_\lambda = \operatorname{mod} S_\lambda / z \mapsto z+1$

$$\frac{q-1}{q^2}\frac{\pi}{\log 2} \le h_\lambda \le \frac{2\pi \operatorname{Re}(\Lambda)}{|\Lambda|^2}$$

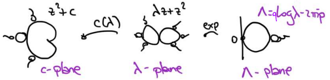
the cylinder contains $q-1$ of the q connected components of the basin of ∞

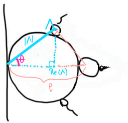
Bers inequality

$$\frac{2\pi \operatorname{Re}(\Lambda)}{|\Lambda|^2} \xrightarrow{\lambda \to \text{root}} \infty,$$

but it is bounded outside the main hyperbolic component:

(on hyp components χ holomorphic)



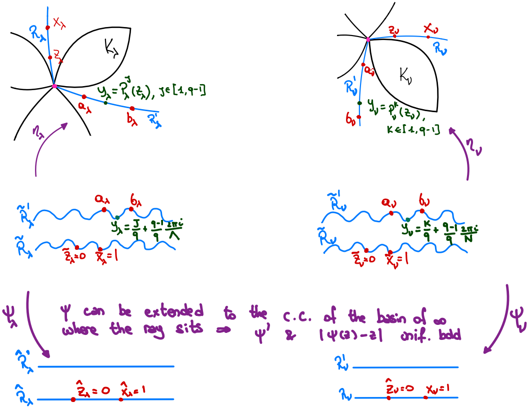


$$\frac{\operatorname{Re}(\Lambda)}{|\Lambda|} = \cos\theta = \frac{|\Lambda|}{\ell}$$

$$\Rightarrow \frac{2\pi \operatorname{Re}(\Lambda)}{|\Lambda|^2} = \frac{2\pi |\Lambda|}{|\Lambda| \ell} = \frac{2\pi}{\ell}$$

$\Rightarrow h_\lambda \le \frac{2\pi}{r}$, $r : \mathbb{D}(r,r) \subset \hat{H}_{p/q}$

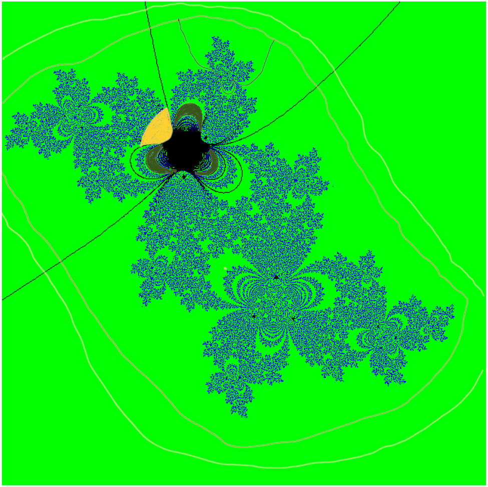
2- Bounding slopes



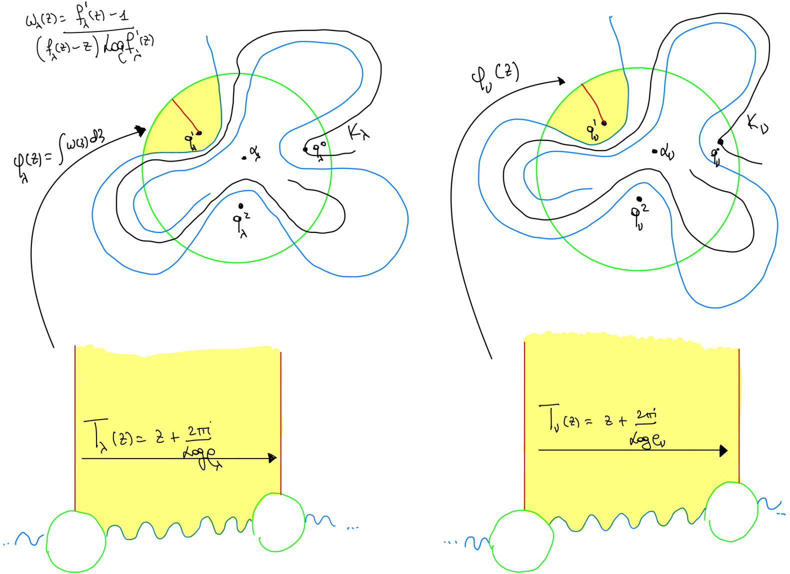
$|a_\lambda - a_\nu| \leq |y_\lambda - y_\nu| + 1 \approx |\frac{2\pi i}{\Lambda} - \frac{2\pi i}{N}|$

1. $|\frac{2\pi i}{\Lambda} - \frac{2\pi i}{N}| \leq |\frac{2\pi i}{\Lambda} - \frac{2\pi i}{\Lambda_0}| + |\frac{2\pi i}{\Lambda_0} - \frac{2\pi i}{N_0}| + |\frac{2\pi i}{N} - \frac{2\pi i}{N_0}|$, with $\lambda_0 \in H_{p/q}$, $\nu_0 \in H_{p'/q}$
2. $|\frac{2\pi i}{\Lambda_0} - \frac{2\pi i}{N_0}|$ uniformly bounded (L-Petersen '17)
3. sublimbs uniformly bounded (Kapiamba)

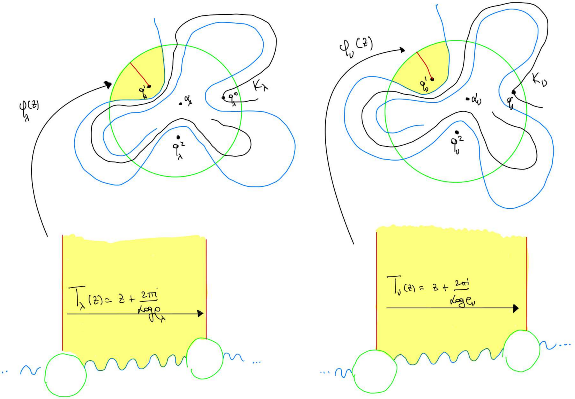
Extension



Extension



Extension



- $T_\lambda \sim T_\nu$:

  1- $\rho_\lambda = f_\lambda(q_\lambda^1)$

  2- $\frac{1}{\log \rho_\lambda} \sim \frac{1}{q\Lambda}$

- Ray stays bounded:

  1- last piece has limit ray landing at parabolic f.p.

  2. In Fatou coordinates heights is bdd (ray invariant by $z \mapsto z+$)

  3. Buff field & Fatou coordinate uniformly close (Petersen-Zakeri)

- Ray in basin of ∞: it cannot get "too close to itself"

Thank you for your attention!