

Gradings and Polynomial Identities in Matrix Algebras

IX Encontro da Pós-Graduação em Matemática da UFBA
- November 2024 -

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In this presentation \mathbb{F} denotes a field.

An **algebra** is a vector space A with a bilinear associative multiplication $A \times A \rightarrow A$, $(a, b) \mapsto ab$.

Examples of algebras:

- An extension of fields $\mathbb{L} \supseteq \mathbb{F}$ is a **commutative** algebra.
- $\mathbb{F}[x_1, \dots, x_n]$ is the algebra of polynomials in the commutative variables x_1, \dots, x_n .
- $M_n(\mathbb{F})$ is an algebra that is not commutative if $n > 1$.
- \mathbb{R}^3 with the cross product \times is not an algebra, the multiplication is not associative.

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The free algebra

We consider the **free algebra** $\mathbb{F}\langle X \rangle$, where $X = \{x_1, x_2, \dots\}$.

As a vector space it has basis consisting of 1 and the words $x_{i_1} \cdots x_{i_n}$ in X .

The multiplication is given by

$$(x_{i_1} \cdots x_{i_n})(x_{j_1} \cdots x_{j_m}) = x_{i_1} \cdots x_{i_n} x_{j_1} \cdots x_{j_m}.$$

If A is an algebra and $x_i \mapsto a_i$ is a map from X to A then there exists a unique homomorphism $\varphi : \mathbb{F}\langle X \rangle \rightarrow A$ that extends it.

Then for $f = f(x_1, \dots, x_n) \in \mathbb{F}\langle X \rangle$ we have $\varphi(f) = f(a_1, \dots, a_n)$.

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We say that $f(x_1, \dots, x_n)$ is a **polynomial identity** for A if

$$f(a_1, \dots, a_n) = 0,$$

for all $a_1, \dots, a_n \in A$. An algebra A is a **p.i.-algebra** if there exists a non-zero polynomial that is an identity for A .

- A is commutative if and only if $[x_1, x_2] := x_1x_2 - x_2x_1$ is a polynomial identity for A ;
- If $\dim A < n$ then

$$s_n(x_1, \dots, x_n) := \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$$

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Examples of p.i.-algebras

- $[[x_1, x_2], x_3]$ is a polynomial identity for the Grassmann algebra;
- $[[x_1, x_2]^2, x_3]$ is a polynomial identity for $M_2(\mathbb{F})$;
- Amitsur and Levitzki (1950) proved that $s_{2n}(x_1, \dots, x_{2n})$ is a polynomial identity for $M_n(\mathbb{F})$;
- $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}]$ is a polynomial identity for $UT_n(\mathbb{F})$;
- Regev (1972) proved that if A and B are p.i.-algebras then $A \otimes B$ is also a p.i.-algebra.

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The set $Id(A)$ of polynomial identities of A is a T -ideal of $\mathbb{F}\langle X \rangle$, i.e., *it is an ideal invariant under the endomorphisms of $\mathbb{F}\langle X \rangle$.*

Given $S \subseteq \mathbb{F}\langle X \rangle$ we denote by $\langle S \rangle_T$ the intersection of all the T -ideals of $\mathbb{F}\langle X \rangle$ that contain S .

If $\langle S \rangle_T = Id(A)$ then we say that S is a T -basis for the T -ideal of identities of A .

Theorem (A. R. Kemer, 1987)

Let \mathbb{F} be a field of characteristic zero. Then every T -ideal of $\mathbb{F}\langle X \rangle$ is finitely based.

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Polynomial identities of $M_2(\mathbb{F})$

Theorem (V. Drensky, 1980)

Let \mathbb{F} be a field of characteristic zero. Then the polynomial identities of $M_2(\mathbb{F})$ follow from the identities

$$s_4(x_1, x_2, x_3, x_4) \text{ and } [[x_1, x_2]^2, x_3].$$

- Let \mathbb{F} be an infinite field with $\text{char } \mathbb{F} \neq 2$. P. Koshlukov (2001) determined a finite basis for $M_2(\mathbb{F})$.
- **Open problem 1:** Does $\text{Id}(M_2(\mathbb{F}))$ admit a finite basis when \mathbb{F} is an infinite field of characteristic 2?
- **Open problem 2:** Determine a finite basis for $M_n(\mathbb{F})$, where $n \geq 3$ and \mathbb{F} is an infinite field.

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Graded algebras

Algebras graded by \mathbb{Z}_2 and their graded polynomial identities appear as a tool in the theory developed by A. Kemer.

Let G be a group and let A be an algebra. A G -grading on A is a decomposition $A = \bigoplus_{g \in G} A_g$ such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$.

Examples of graded algebras:

- Let G be a group with neutral element e and let A be an algebra. Then $A = \bigoplus_{g \in G} A_g$, where $A_e = A$ and $A_g = 0$ for all $g \neq e$, is a G -grading on A .
- Let $R = M_2(\mathbb{F})$. Then $R = R_0 \oplus R_1$, where

$$R_0 = \begin{pmatrix} \mathbb{F} & 0 \\ 0 & \mathbb{F} \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 0 & \mathbb{F} \\ \mathbb{F} & 0 \end{pmatrix}.$$

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- We assume that $\text{char } \mathbb{F} \neq 2$ and let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $A^2 = B^2 = I$ and $BA = -AB$. Hence $D = M_2(\mathbb{F})$ admits a G -grading $D = \bigoplus_{g \in G} D_g$, where

$$D_{(0,0)} = \mathbb{F}I, \quad D_{(1,0)} = \mathbb{F}A, \quad D_{(0,1)} = \mathbb{F}B \quad \text{and} \quad D_{(1,1)} = \mathbb{F}AB.$$

Let A and B be G -graded algebras. A homomorphism $\varphi : A \rightarrow B$ is a **homomorphism of graded algebras** if $\varphi(A_g) \subseteq B_g$, for all $g \in G$. If φ is an isomorphism we say that the graded algebras A and B are **isomorphic**.

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$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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$$D_{(0,0)} = \mathbb{F}I, \quad D_{(1,0)} = \mathbb{F}A, \quad D_{(0,1)} = \mathbb{F}B \quad \text{and} \quad D_{(1,1)} = \mathbb{F}AB.$$

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Graded polynomial identities

Let $X_G = \dot{\cup}_{g \in G} X_g$, where $X_g = \{x_{1,g}, x_{2,g}, \dots\}$. The free algebra $\mathbb{F}\langle X_G \rangle$ admits the following G -grading: $\mathbb{F}\langle X_G \rangle = \bigoplus_{g \in G} \mathbb{F}\langle X_G \rangle_g$, where

$$\mathbb{F}\langle X_G \rangle_g = \text{span}\{x_{i_1, g_1} \cdots x_{i_n, g_n} \mid g_1 \cdots g_n = g\}.$$

$\mathbb{F}\langle X_G \rangle$ with this grading is the free G -graded algebra.

We say that $f(x_{1,g_1}, \dots, x_{n,g_n})$ is a **graded polynomial identity** for A if

$$f(a_{1,g_1}, \dots, a_{n,g_n}) = 0,$$

for all $a_{1,g_1} \in A_{g_1}, \dots, a_{n,g_n} \in A_{g_n}$.

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The set $Id_G(A)$ of graded polynomial identities of $A = \bigoplus_{g \in G} A_g$ is a T_G -ideal of $\mathbb{F}\langle X_G \rangle$, i.e., it is an ideal invariant under the graded endomorphisms of $\mathbb{F}\langle X_G \rangle$.

Given $S \subseteq \mathbb{F}\langle X_G \rangle$ we denote by $\langle S \rangle_{T_G}$ the intersection of all the T_G -ideals of $\mathbb{F}\langle X_G \rangle$ that contain S .

If $\langle S \rangle_{T_G} = Id_G(A)$ then we say that S is a **basis** for the T_G -ideal of identities of A .

Theorem (E. Aljadeff, A. Kanel-Belov; I. Sviridova, 2010)

Let \mathbb{F} be a field of characteristic zero and let G be a finite group. Let A be a p.i.-algebra graded by G . Then the T_G -ideal $Id_G(A)$ is finitely based.

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Graded identities for $M_2(\mathbb{F})$

Let $R = M_2(\mathbb{F}) = R_0 \oplus R_1$ where

$$R_0 = \begin{pmatrix} \mathbb{F} & 0 \\ 0 & \mathbb{F} \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 0 & \mathbb{F} \\ \mathbb{F} & 0 \end{pmatrix}.$$

Theorem (O. M. Di Vincenzo; P. Koshlukov, S. Sardinha, 2002)

Let \mathbb{F} be an infinite field of characteristic different from 2. Then the \mathbb{Z}_2 -graded identities of $R = M_2(\mathbb{F})$ follow from the graded identities $[x_{1,0}, x_{2,0}]$ and $x_{1,1}x_{2,1}x_{3,1} - x_{3,1}x_{2,1}x_{1,1}$.

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Gradings on matrix algebras

We consider the following problem:

Problem

Describe the group gradings on $M_n(\mathbb{F})$. For a G -grading $M_n(\mathbb{F}) = R = \bigoplus_{g \in G} R_G$ determine a basis for $\text{Id}_G(R)$.

A general version of the first part of this problem (that was posed to A. V. Kelarev by Zelmanov) appears in the section "Open Questions" of [3].

Elementary gradings on matrix algebras

Let $A = M_n(\mathbb{F})$. We denote by $E_{i,j}$ the elementary matrix with 1 in the (i, j) -th position and 0 elsewhere. Then

$$E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l},$$

where $\delta_{j,k}$ is the Kronecher delta.

Let G be a group and let $\mathbf{g} = (g_1, \dots, g_n) \in G^n$. The decomposition $A = \bigoplus_{g \in G} A_g$, where

$$A_g = \text{span}\langle E_{i,j} \mid g_i g_j^{-1} = g \rangle,$$

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This is the elementary grading determined by \mathbf{g} .

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Pauli gradings

We assume that \mathbb{F} contains a primitive n -th root of unity ϵ . Let $G = \mathbb{Z}_n \times \mathbb{Z}_n$. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} \epsilon^{n-1} & 0 & \cdots & 0 \\ 0 & \epsilon^{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then $A^n = B^n = I$ and $BA = \epsilon AB$, hence we obtain the division G -grading $R = M_n(\mathbb{F}) = \bigoplus_{g \in G} R_g$ where

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We refer to this as the Pauli grading on $M_n(\mathbb{F})$.

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Graded identities for matrix algebras

Let G be a group and let $R = \bigoplus_{g \in G} R_g$ be the algebra $M_n(\mathbb{F})$ with an elementary G -grading induced by an n -tuple $\mathbf{g} \in G^n$.

Assume that the entries of \mathbf{g} are pairwise distinct (equiv. R_e is commutative).

The polynomials

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$$x_{1,g}x_{2,g^{-1}}x_{3,g} - x_{3,g}x_{2,g^{-1}}x_{1,g} \text{ if } g \neq e \text{ and } R_g \neq 0 \tag{2}$$

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Basis for the graded identities of matrix algebras

Vasilovsky proved in 1998 that if $\text{char } \mathbb{F} = 0$ then a basis for the graded identities of $M_n(\mathbb{F})$ with its canonical \mathbb{Z} -gradings consists of the polynomials (1), (2) and (3). He also proved in 1999 that a basis for the \mathbb{Z}_n -graded identities of $M_n(\mathbb{F})$ consists of (1) and (2).

Sardinha proved in 2002 and 2003 that these results remain valid if \mathbb{F} is an infinite field.

Bahturin and Drensky proved in 2002, assuming that $\text{char } \mathbb{F} = 0$ and that the neutral component is commutative, that a basis for the graded identities of $M_n(\mathbb{F})$ with an elementary grading consists of (1), (2), (3) and a finite number of monomial identities of degree bounded by a function $f(n)$.

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Basis for the graded identities of block-triangular matrices

Theorem (Centrone, Mello, D.D - 2022)

Let \mathbb{F} be an infinite field. Let G be any group and let $\mathbf{g} = (g_1, \dots, g_n) \in G^n$ induce an elementary G -grading of $M_n(\mathbb{F})$ where the elements g_1, \dots, g_n are pairwise different. Then a basis of the graded polynomial identities of $M_n(\mathbb{F})$ consists of (1)-(3) and a finite number of identities of the form $x_{i_1, g_1} \dots x_{i_k, g_k}$, where $2 \leq k \leq n$.

Induced gradings on tensor products

Let $A = M_n(\mathbb{F}) = \bigoplus_{g \in G} A_g$ be the matrix algebra with the elementary grading determined by $\mathbf{g} = (g_1, \dots, g_n) \in G^n$.

Given a G -graded algebra $B = \bigoplus_{g \in G} B_g$, the algebra $R = A \otimes B$ has a G -grading $R = \bigoplus_{g \in G} R_g$, where

$$R_g = \text{span}\langle E_{ij} \otimes b \mid b \in B_h, g_i h g_j^{-1} = g \rangle.$$

These gradings are called **induced gradings** and were introduced by Yu. A. Bahturin and M. Zaicev in their classification of group gradings on matrix algebras.

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Let $A = M_n(\mathbb{F}) = \bigoplus_{g \in G} A_g$ be the matrix algebra with the elementary grading determined by $\mathbf{g} = (g_1, \dots, g_n) \in G^n$.

Given a G -graded algebra $B = \bigoplus_{g \in G} B_g$, the algebra $R = A \otimes B$ has a G -grading $R = \bigoplus_{g \in G} R_g$, where

$$R_g = \text{span}\langle E_{ij} \otimes b \mid b \in B_h, g_i h g_j^{-1} = g \rangle.$$

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Graded Wedderburn-Artin Theorem

Theorem

If R is graded simple of finite dimension then there exist a graded division algebra D and $\mathbf{g} \in G^t$ such that R is isomorphic to $M_t(\mathbb{F}) \otimes_{\mathbb{F}} D$ with the grading induced by \mathbf{g} .

To classify group gradings on matrix algebras it is sufficient to classify division gradings on these algebras.

If p is a prime number then a grading on $M_p(\mathbb{F})$ is either an elementary grading or a division grading.

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Gradings on Matrix Algebras

In 2003 R. Khazal and C. Boboc provided a complete answer to Zelmanov's problem for the algebra $M_2(\mathbb{F})$, for any field \mathbb{F} and any group G .

We point out that the types of gradings on $M_2(\mathbb{F})$ are related to the characteristic of the field \mathbb{F} .

They proved that any such grading reduces to a \mathbb{Z}_2 -grading, a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading or to an elementary grading.

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Gradings on matrix algebras of prime order

Assume that $\text{char } \mathbb{F} \neq p$ and that \mathbb{F} contains a primitive p -th root of unit ϵ .

We construct a grading on $M_p(\mathbb{F})$ by a group G with support $T = \langle t \rangle$, where t is an element of G of order p .

Let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a & 0 & \cdots & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} \epsilon^{p-1} & 0 & \cdots & 0 \\ 0 & \epsilon^{p-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad (4)$$

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Then we have

$$BA = \epsilon AB, \quad A^p = aI \quad \text{and} \quad B^p = I.$$

The matrices $A^i B^j$, where $0 \leq i, j < p$ form a basis for $M_p(\mathbb{F})$.

Then we obtain a grading on $M_p(\mathbb{F})$ denoted $D_{a,\epsilon,t}$ on where A is homogeneous of degree e and B is homogeneous of degree t .

We can prove that $D_{a,\epsilon,t}$ is a division grading on $M_p(\mathbb{F})$ with support $T = \langle t \rangle$.

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where a is an element of $\mathbb{F} \setminus \{b^p - b \mid b \in \mathbb{F}\}$.

Note that A is the companion matrix of the polynomial $x^p - x - a$ and B is the upper triangular matrix with (i, j) -th entry equal to $\binom{j-1}{i-1}$ whenever $j \geq i$.

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Then we have

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Gradings on matrix algebras of prime order

Theorem (D. D., A. Guimarães, 2024)

Let G be a group, \mathbb{F} be a field and p be a prime number. We assume that \mathbb{F} contains a primitive p -th root of unit if $\text{char } \mathbb{F} \neq p$. Then if $\text{char } \mathbb{F} \neq p$ any G -grading on $M_p(\mathbb{F})$ is isomorphic to one of the following types:

- 1 An elementary G -grading.
- 2 A cyclic division grading $D_{a,\epsilon,t}$ where $a \in \mathbb{F} \setminus \mathbb{F}^p$, ϵ is a primitive p -th root of unit and t is an element of G of order p .
- 3 A fine division grading $D = \bigoplus_{g \in T} D_g$ with support $T = \langle t \rangle \times \langle u \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$ such that $D_{t^i u^j} = \mathbb{F} A^i B^j$ for $0 \leq i, j < p$, here A, B are invertible matrices such that $A^p, B^p \in \mathbb{F}I$ and $AB = \epsilon BA$, where ϵ is a primitive p -th root of unity.

If $\text{char } \mathbb{F} = p$ then every grading on $M_p(\mathbb{F})$ is isomorphic an elementary G -grading or a cyclic division grading $D_{a,t}$ where $a \in \mathbb{F} \setminus \{b^p - b \mid b \in \mathbb{F}\}$ and t is an element of G of order p .

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Thank you!