A functional Central Limit theorem for the general Brownian motion on the half-line

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Joint work with:

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- For each $n \in \mathbb{N}$ consider Banach spaces B_n and B.

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- Acting on these spaces, consider $\{\mathsf{T}_n(t) : t \geq 0\}$ and $\{\mathsf{T}(t) : t \geq 0\}$ strongly continuous contraction semigroups, with generators L_n and L respectively. Also, denote by $\mathfrak{D}(\mathsf{L})$ the domain where the generator L is defined, as well for $\mathfrak{D}(\mathsf{L}_n)$.

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- Acting on these spaces, consider $\{\mathsf{T}_n(t) : t \geq 0\}$ and $\{\mathsf{T}(t) : t \geq 0\}$ strongly continuous contraction semigroups, with generators L_n and L respectively. Also, denote by $\mathfrak{D}(\mathsf{L})$ the domain where the generator L is defined, as well for $\mathfrak{D}(\mathsf{L}_n)$.
- Finally, let $\pi_n, \Xi_n : \mathsf{B} \to \mathsf{B}_n$ linear operators.

π_n are the natural projections while Ξ_n will play an important role ensuring convergence.

By $\mathfrak{D}(\mathsf{L}^2)$ we mean that $f, \mathsf{L}f \in \mathfrak{D}(\mathsf{L})$. Define by $\Phi_n := \pi_n + \Xi_n$

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(A) If $f \in \mathsf{B}$ then $\Phi f \in \mathfrak{D}(\mathsf{L}^2)$,

(B) There exist sequences $s_n^1 \downarrow 0$, $s_n^2 \downarrow 0$ and $s_n^3 \downarrow 0$ such that, for any $f \in \mathfrak{D}(\mathsf{L}^2)$,

$$\|\pi_n \mathsf{L} f - \mathsf{L}_n \Phi_n f\| \leq s_n^1 \|f\| + s_n^2 \|\mathsf{L} f\| + s_n^3 \left\|\mathsf{L}^2 \mathsf{f}\right\| .$$

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(C) There exist sequences $r_n^1 \downarrow 0$ and $r_n^2 \downarrow 0$ such that, $f \in \mathfrak{D}(\mathsf{L}^2)$,

$$\|\Xi_n f\| \leq r_n^1 \|f\| + r_n^2 \|\mathsf{L} f\| .$$

Theorem

Under hypotheses $A - C$*, for any* $f \in \mathfrak{D}(\mathsf{L}^2)$ *and for each* t *in a compact interval* $[0, b]$ *we have that*

$$\|\mathsf{T}_n(t)\pi_n f - \pi_n \mathsf{T}(t) f\| \lesssim \max\{s_n^1, r_n^1\}\,\|f\| + \max\{s_n^2, r_n^1, r_n^2\}\,\|\mathsf{L}f\| + \max\{s_n^3, r_n^2\}\,\left\|\mathsf{L}^2 f\right\|$$

By $f \lesssim g$ we mean that there exists a cosntant $c > 0$ such that $f \leq cg$ in the hole space

Let (S, d) be a Polish space (separable completely metrizable topological space). By $\mathcal{C}_0(\mathsf{S})$ we denote the space of continuous functions vanishing at infinity, i.e., fixed any $x_0 \in \mathsf{S}$

$$\lim_{d(x,x_0)\to\infty} f(X) = 0\,.$$

(I) the Banach space $\mathcal{C}_0(\mathsf{S})$ is equipped with the uniform topology, π_n is solely the restriction of S to a subset S_n, i.e., $\pi_n f = f|_{\mathsf{S}_n}$

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(II) There is a sequence $\{f_k : k \geq 0\} \subset \mathcal{C}_0(\mathsf{S})$ such that $\mathrm{span}(\{f_k\})$ is dense. Additionaly, for every k, there exists $\{f_{j,k}\} \subset \mathfrak{D}(\mathsf{L}^2)$ satisfying that $\|f_{j,k} - f_k\| \to 0$ uniformly as $j \to \infty$,

(III) For every j, k, it holds that

$$\|\mathsf{L} f_{j,k}\| \leq h_j^1 \|f_k\| \qquad \text{and} \qquad \left\|\mathsf{L}^2 f_{j,k}\right\| \leq h_2^j \|f_k\| \,,$$

where the sequences h_j^1 and h_j^2 satisfies $\sum_{j\geq 0} 2^{-j} h_j^i < \infty$ for $i = 1, 2$. Moreover, $\sum_{k\geq 0} 2^{-k} \|f_k\| < \infty$ and $\sum_{j,k\geq 0} 2^{-j-k} \|f_{j,k}\| < \infty$.

(IV) The semigroup T associated to the generator L is Lipschitz in the sense that, for each $t \geq 0$, there exists a constant $M = M(t)$ such that

$$|\mathsf{T}(t) f(x) - \mathsf{T}(t) f(y)| \leq M \cdot \|f\| \cdot d(x, y)$$

Let us equipp the space of sub-probability measures with a suitable distance that imples vague convergence.

Definition

Let $\{f_k : k \geq 0\} \subset \mathcal{C}_0(\mathsf{S})$ and $\{f_{j,k} : j \geq 0\} \subset \mathfrak{D}(\mathsf{L}^2)$ be as in II. We define

$$\mathbf{d}(\mu, \nu) := \sum_{j,k \geq 0} \frac{1}{2^{j+k}} \left(\left| \int f_{j,k}\, d\mu - \int f_{j,k}\, d\nu \right| \wedge 1 \right)$$

for any $\mu, \nu \in \mathcal{M}_{\leq 1}(\mathsf{S})$.

Theorem

Assume hypotheses $A - C$ and $I - IV$, Let $\{X(t) : t \in [0, T]\}$ and $\{X_n(t) : t \in [0, T]\}$ be the feller processes on S and S_n associated to L and L_n respectively. Assume that the process starts at $x_n \in \mathsf{S}_n$ and $x \in \mathsf{S}$ and $d(x, x_n) \leq i_n$ for some $i_n \downarrow 0$. Fix some $t > 0$. Denote by μ and μ_n the probability distributions on S and S_n induced by $X(t)$ and $X_n(t)$ respectively starting from the points x and x_n. then

$$\mathbf{d}(\mu, \mu_n) \lesssim \max\{i_n, s_n^1, s_n^2, s_n^3, r_n^1, r_n^2\} .$$

Moreover, we also have pathwise convergence $X_n \Rightarrow X$ in the Skorohod space $\mathsf{D}_{\mathsf{S}}[0, \infty)$.

Definition

A general Brownian motion on the positive halk-line is a difussion process W on the set $\mathbb{G} := \{\Delta\} \cup [0, \infty)$ such that the absorbed process $\{\mathsf{W}(t \vee T_0) : t \geq 0\}$ on $[0, \infty)$ has the same distribution as B_0 for any startint point $x \geq 0$.

In the definition above, by T_0 we mean the hitting time of zero and B_0 denotes the absorbed BM.

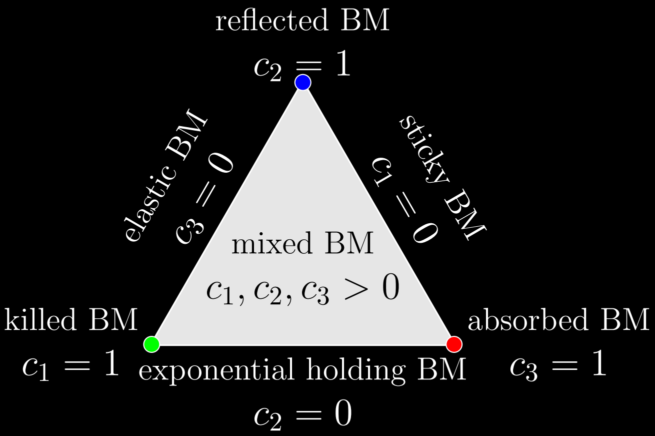
Theorem

Any general Brownian motion W *on* $[0,\infty)$ *has generator* $\mathsf{L} = 1/2\Delta$ *with corresponding domain*

$$\mathfrak{D}(\mathsf{L}) := \{f \in \mathcal{C}_0^2(\mathbb{G}) : f'' \in \mathsf{C}_0(\mathbb{G}), c_1 f(0) - c_2 f'(0) + c_3/2 f''(0) = 0\}.$$

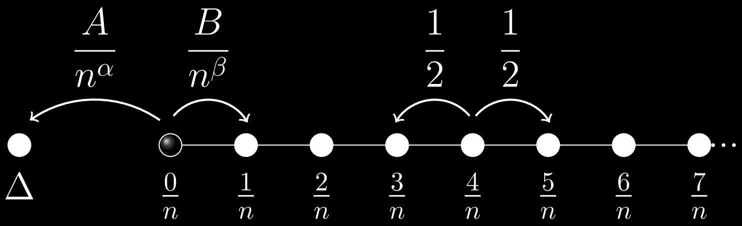
for some $c_i \geq 0$ *such that* $c_1 + c_2 + c_3 = 1$ *and* $c_1 \neq 1$*.*

GMB characterization.



Figur: Description of the general Brownian motion on the half-line according to the chosen values of $c_1, c_2, c_3 \geq 0$ on the simplex $c_1 + c_2 + c_3 = 1$. Note that the killed BM, which formally corresponds to $c_1 = 1$, is not rigorously a case of the general BM.

Boundary Random Walk



Generator of boundary random walk.

$$\mathsf{L}_n f(x) = \begin{cases} \frac{1}{2}\Big[f\big(x+\tfrac{1}{n}\big) - f(x)\Big] + \frac{1}{2}\Big[f\big(x-\tfrac{1}{n}\big) - f(x)\Big], \ x = \tfrac{1}{n}, \tfrac{2}{n}, \dots \\ \frac{A}{n^\alpha}\Big[f\big(\Delta\big) - f\big(\tfrac{0}{n}\big)\Big] + \frac{B}{n^\beta}\Big[f\big(\tfrac{1}{n}\big) - f\big(\tfrac{0}{n}\big)\Big], \quad \text{for } x = \tfrac{0}{n}. \end{cases}$$

Fix $u, t > 0$. Let $\{X_n(t) : t \geq 0\}$ be the boundary random walk of parameters $\alpha, \beta, A, B \geq 0$ sped up by n^2 (that is, whose generator is $n^2 \mathsf{L}_n$), starting from the point $\frac{\lfloor un \rfloor}{n} \in \mathbb{G}_n \subset \mathbb{G}$ and denote by $\mu_n = \mu_n(t)$ the distribution of X_n at time $t > 0$. Recall the metric $\mathbf{d}$, and denote by $\mu = \mu(t)$ the distribution at time $t > 0$ of the limit process in each of following cases.

Theorem

Suppose the technical conditions. Then

1. *If* $\alpha = \beta + 1$ *and* $\beta \in [0,1)$, *then* $\{X_n(t) : t \geq 0\}$ *converges weakly to* $\{X^{EBM}(t) : t \geq 0\}$ *in the* J_1*-Skorohod topology of* $\mathsf{D}_{\mathbb{G}}[0,\infty)$, *where* X^{EBM} *is the elastic BM on* $\mathbb{G} = \{\Delta\} \cup \mathbb{R}_{\geq 0}$ *of parameters*

$$c_1 = \frac{B}{A+B}, \quad c_2 = \frac{A}{A+B} \quad \text{and} \quad c_3 = 0$$

starting from the point u. *Moreover,*

1.1 *if* $\beta \in (0,1)$, *then* $\mathbf{d}(\mu_n, \mu) \lesssim \max\{n^{-\beta}, n^{\beta-1}\}$,

1.2 *if* $\beta = 0$, *then* $\mathbf{d}(\mu_n, \mu) \lesssim n^{-1}$.

Theorem

2 *If* $\alpha \in (2, +\infty]$ *and* $\beta = 1$, *then* $\{X_n(t) : t \geq 0\}$ *converges weakly to* $\{X^{SBM}(t) : t \geq 0\}$ *in the* J_1*-Skorohod topology of* $\mathsf{D}_{\mathbb{G}}[0, \infty)$, *where* X^{SBM} *is the sticky BM on* $\mathbb{R}_{\geq 0}$ *of parameters*

$$c_1 = 0, \quad c_2 = \frac{B}{B+1}, \quad \textit{and} \quad c_3 = \frac{1}{B+1}$$

starting from the point u. *Moreover, in this case,*

$$\mathbf{d}(\mu_n, \mu) \lesssim \max\{n^{2-\alpha}, n^{-1}\}\,.$$

Theorem

3 *If* $\alpha = 2$ *and* $\beta \in (1, +\infty]$, *then* $\{X_n(t) : t \geq 0\}$ *converges weakly to* $\{X^{EHBM}(t) : t \geq 0\}$ *in the* J_1*-Skorohod topology of* $\mathsf{D}_{\mathbb{G}}[0, \infty)$, *where* X^{EHBM} *is the exponential holding BM on* $\mathbb{G} = \{\Delta\} \cup \mathbb{R}_{\geq 0}$ *of parameters*

$$c_1 = \frac{A}{1+A}, \quad c_2 = 0 \quad \textit{and} \quad c_3 = \frac{1}{1+A}$$

starting from the point u. *Moreover, for* $\beta \in (2, \infty)$,

$$\mathbf{d}(\mu_n, \mu) \lesssim \max\{n^{2-\beta}, n^{-1}\}.$$

Theorem

4 *If* $\alpha > \beta + 1$ *and* $\beta \in [0, 1)$, *then* $\{X_n(t) : t \geq 0\}$ *converges weakly to* $\{X^{RBM}(t) : t \geq 0\}$ *in the* J_1*-Skorohod topology of* $\mathsf{D}_{\mathbb{R}_{\geq 0}}[0, \infty)$, *where* X^{RBM} *is the reflected BM on* $\mathbb{R}_{\geq 0}$, *of parameters*

$$c_1 = 0, \quad c_2 = 1 \quad \textit{and} \quad c_3 = 0$$

starting from the point u. *Moreover, for* $\beta \in (0, 1)$,

4.1 *if* $1 + \beta < \alpha < 2$, *then* $\mathbf{d}(\mu_n, \mu) \lesssim \max\{n^{-\beta}, n^{-\alpha+\beta+1}, n^{\alpha-2}\}$,

4.2 *if* $\alpha = 2$, *then* $\mathbf{d}(\mu_n, \mu) \lesssim \max\{n^{-\beta}, n^{\beta-1}\}$,

4.3 *if* $\alpha > 2$, *then* $\mathbf{d}(\mu_n, \mu) \lesssim \max\{n^{2-\alpha}, n^{-\beta}, n^{\beta-1}\}$.

Theorem

5 *If* $\alpha \in (2, \infty]$ *and* $\beta \in (1, +\infty]$, *then* $\{X_n(t) : t \geq 0\}$ *converges weakly to* $\{X^{ABM}(t) : t \geq 0\}$ *in the* J_1*-Skorohod topology of* $\mathsf{D}_{\mathbb{R}_{\geq 0}}[0, \infty)$, *where* X^{ABM} *is the absorbed BM on* $\mathbb{R}_{\geq 0}$, *of parameters*

$$c_1 = 0, \quad c_2 = 0 \quad \text{and} \quad c_3 = 1$$

starting from the point u. *Moreover, for* $\alpha > 2$ *and* $\beta > 2$,

$$\mathbf{d}(\mu_n, \mu) \lesssim \max\{n^{2-\alpha}, n^{2-\beta}, n^{-1}\}.$$

Theorem

6 *If* $\alpha = 2$ *and* $\beta = 1$, *then* $\{X_n(t) : t \geq 0\}$ *converges weakly to* $\{X^{MBM}(t) : t \geq 0\}$ *in the* J_1*-Skorohod topology of* $\mathsf{D}_{\mathbb{G}}[0, \infty)$, *where* X^{MBM} *is the mixed BM on* $\mathbb{G} = \{\Delta\} \cup \mathbb{R}_{\geq 0}$ *of parameters*

$$c_1 = \frac{A}{1 + A + B}, \quad c_2 = \frac{B}{1 + A + B} \quad \text{and} \quad c_3 = \frac{1}{1 + A + B}$$

starting from the point u. *Moreover,* $\mathbf{d}(\mu_n, \mu) \lesssim n^{-1}$.

Since the natural state space of the killed BM is $\mathbb{G} = \{\Delta\} \cup (0, \infty)$, we need a different setup. Let $\tau_n : \mathbb{G} \to \mathbb{G}$ be the shift to the right of $1/n$ given by

$$\tau_n(\Delta) = \Delta \quad \text{and} \quad \tau_n(u) = u + \frac{1}{n} \quad \text{for } u \in [0, \infty)\,.$$

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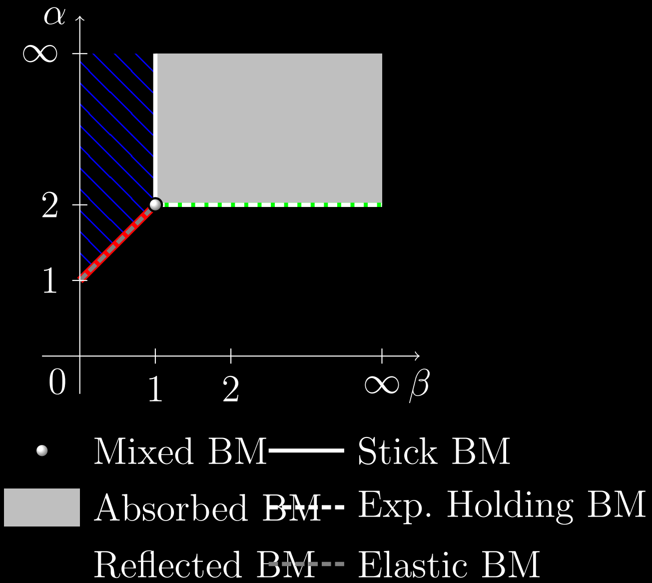
$$\tau_n(\Delta) = \Delta \quad \text{and} \quad \tau_n(u) = u + \tfrac{1}{n} \quad \text{for } u \in [0, \infty) .$$

Theorem

If $\alpha < 1 + \beta$ for $\beta \in [0, 1]$, or $\beta > 1$, then $\{\tau_n X_n(t) : t \geq 0\}$ converges weakly to $\{X^{KBM}(t) : t \geq 0\}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{G}}[0, \infty)$, where in this case X^{KBM} is the killed BM on $\mathbb{G} = \{\Delta\} \cup (0, +\infty)$, which is formally the general BM of parameters

$$c_1 = 1, \quad c_2 = 0 \quad \textit{and} \quad c_3 = 0 .$$

Phase transition diagram.



References

- Erhard, D., Franco, T., Jara, M., and Pimenta, E. (2024). A Functional Central Limit Theorem for the General Brownian Motion on the Half-Line. arXiv preprint arXiv:2408.06830.

Thank you.

SCIENDI FUNDAMEN

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