Sobre a dimensão fractal de atratores em equações de evolução

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Salvador - BA, 21 de Novembro de 2024

EPGMAT IX Encontro da Pós-Graduação em Matemática da UFBA

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Semigroups (or Dynamical Systems)

- Let $(X, \|\cdot\|_X)$ be a Banach space and $S(t) : X \longrightarrow X$ be an application for each $t \geq 0$.

- We say that a family $\{S(t) : t \geq 0\} \subset \mathcal{C}(X)$ is a semigroup if
  i) $S(0) = Id_X$;
  ii) $S(t+s) = S(t)S(s)$, for all $t, s \geq 0$;
  iii) The map $S : X \times [0, \infty) \longrightarrow X$ given by $S(t)x$ is continuous.

How to obtain dynamical systems?

Suppose that

$$\begin{cases} \dfrac{d}{dt}u(t,x) = F\big(u(t,x)\big), \qquad t > 0, \qquad x \in \Omega \subset \mathbb{R}^N \\ u(0,x) = u_0(x) \in X \end{cases}$$

has a unique global solution $u(t)$ for each initial data $u_0 \in X$ which is continuous with respect to the initial data.

Defining $S(t) : X \longrightarrow X$ by

$$S(t)u_0 := u(t, 0; u_0)$$

we have that $\{S(t) : t \geq 0\}$ is a semigroup.

Global attractors

- A set $\mathcal{A} \subset X$ is the global attractor for $\left\{S(t)\right\}_{t\geq 0}$ if

  i) $\mathcal{A}$ is compact;

  ii) $\mathcal{A}$ is invariant by $\{S(t)\}_{t\geq 0}$, i.e.,

  $$S(t)\mathcal{A} = \mathcal{A}, \qquad \forall t \geq 0.$$

  iii) $\mathcal{A}$ is an attracting set for $\{S(t)\}_{t\geq 0}$, i.e., for all $D \subset X$ bounded we have

  $$\lim_{t\to\infty} \mathrm{dist}_H\big(S(t)D, \mathcal{A}\big) = 0,$$

  where

  $$\mathrm{dist}_H(B, A) := \sup_{b\in B} \inf_{a\in A} \|b - a\|_X.$$

The discrete setting

- A set $\mathcal{A} \subset X$ is the global attractor for a map $S : X \longrightarrow X$ if

  i) $\mathcal{A}$ is compact.

  ii) $\mathcal{A}$ is invariant by S, i.e., $S(\mathcal{A}) = \mathcal{A}$.

  iii) $\mathcal{A}$ is an attracting set for S, i.e., for all $D \subset X$ bounded we have

$$\lim_{n \to \infty} \mathrm{dist}_H\big(S^n(D), \mathcal{A}\big) = 0.$$

- If $S := S(t_0)$ for some t_0, the global attractors of S and $\{S(t)\}_{t \geq 0}$ coincide.

Motivation

- Question 1: If $\mathcal{A}$ is the attractor of a compact map S, is it a finite-dimensional set?

- Question 2: Among the methods to estimate the fractal dimension of attractors, is it possible to make a comparison (in some sense) between them?

A bit of the path to get some answers for last questions...

- It started at ICMC - USP in 2016 when I first had to choose the problem I would investigate in my PhD.

ICMC and Alexandre Nolasco de Carvalho (Advisor)

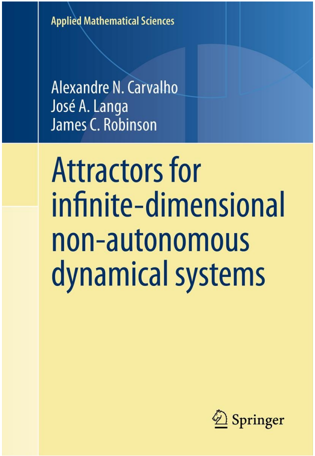
- In 2018 I arrived to Seville and discussed with Alexandre and Langa, obtaining the first partially answers for the problem.

Seville-Spain and José Langa (Coadvisor)

- Yet in 2018 I met James Robinson at the University of Seville, one of the greatest researchers in dimension theory.

James Robinson

- They were the writers of my main reference for the thesis...



- In 2019 I went to Coventry-UK and we finally finished this work.

Motivation

- Question 1: If $\mathcal{A}$ is the attractor of a compact map S, is it a finite-dimensional set?

- Question 2: Among the methods to estimate the fractal dimension of attractors, is it possible to make a comparison (in some sense) between them?

About Question 1

- Answer 1: In general, no!!

For example the map $S : \ell^2 \to \ell^2$ given by

$$(S\mathbf{x})_j = \begin{cases} j^{-1}\frac{x_j}{|x_j|} & , |x_j| > j^{-1} \\ x_j & , |x_j| \leq j^{-1}. \end{cases}$$

This map is compact, but its attractor is the set

$$\begin{aligned} \mathcal{A} &= \{\mathbf{x} \in \ell^2 : \ |x_j| \leq 1/j\} \\ &= [-1, 1] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{3}, \frac{1}{3}\right] \times \cdots \end{aligned}$$

which is an infinite-dimensional subset of ℓ^2.

Going a bit deeper on the problem

However, the answer to this question is *yes* if as well as being compact S is differentiable:

$$S \text{ compact} + \text{differentiable} \implies DS \text{ compact} \overset{\text{Mañé}}{\implies} \dim_F(\mathcal{A}) < \infty.$$

R. Mañé, *On the dimension of the compact invariant sets of certain non-linear maps*. Lecture Notes in Mathematics **898**, Springer-Verlag, New York, pp. 230-242, 1981.

A.N. Carvalho, J.A. Langa & J.C. Robinson, *Finite-dimensional global attractors in Banach spaces*. Journal of Differential Equations **249**, pp. 3099-3109, 2010.

Theorem (New proof by Carvalho, Cunha, Langa, Robinson - JMAA, 2022)

Let X be a Banach space and $\mathcal{A}$ a compact subset of X that is invariant for a continuously differentiable map $S : X \to X$. Suppose that

$$DS(x) = C_x + L_x, \qquad x \in \mathcal{A},$$

where

- C_x *and* L_x *are both linear,*
- $C_x : X \to X$ *is compact for each* $x \in \mathcal{A}$,
- C_x *is continuous in* x *(on* $\mathcal{A}$*),*
- *there exists* $0 < \lambda < 1/4$ *such that* $\|L_x\|_{\mathcal{L}(X)} \leq \lambda$ *for every* $x \in \mathcal{A}$.

Then $\dim_F(\mathcal{A}) < \infty$.

The first results on the dimensionality of global attractors

John Mallet-Paret and Ricardo Mañé

Defining the terms

- Let $(X, \|\cdot\|_X)$ be a Banach space and $\mathcal{A} \subseteq X$ be a compact set.
- Given $\varepsilon > 0$,

$$N_X[\mathcal{A};\varepsilon]$$

denotes the minimum number of ε-balls with centres at points of $\mathcal{A}$ which are necessary to cover $\mathcal{A}$.

- The **fractal dimension** of $\mathcal{A}$ is defined as

$$\dim_F(\mathcal{A}) := \limsup_{\varepsilon \to 0^+} \frac{\log N_X[\mathcal{A};\varepsilon]}{\log 1/\varepsilon}.$$

But why the fractal dimension?

- One possible answer can be:

Embedding Theorem:

"If

$$\dim_F(\mathcal{A}) < \frac{n}{2},$$

then $\mathcal{A}$ can be embedded into $\mathbb{R}^n$."

(Mañé (1981), Hunt/Kaloshin (1999), Robinson (2009) , ...)

Obtaining estimates: smoothing method

- Let Z be a Banach space compactly embedded in X.
- Let
$$N_\varepsilon := N_X[B_Z(0,1);\varepsilon].$$

Theorem (Carvalho, Cunha, Langa, Robinson - JMAA, 2022)

Let Z and X be Banach spaces with Z compactly embedded in X and $\mathcal{A}$ a compact subset of X that is invariant for a continuously differentiable map $S : X \to X$. Suppose that

$$DS(x) = C_x + L_x, \qquad x \in \mathcal{A},$$

where

- $C_x \in \mathcal{L}(X, Z)$ *and* $L_x \in \mathcal{L}(X)$,
- C_x *is continuous in* x *(on* $\mathcal{A}$*),*

Continue...

Theorem (...)

- $C_x : X \to Z$ satisfies the smoothing property for each $x \in \mathcal{A}$, i.e., there is $\kappa > 0$ such that

$$\|C_x u\|_Z \leq \kappa \|u\|_X, \qquad \text{for all } u \in X,$$

- there exists $0 < \lambda < 1/4$ such that

$$\|L_x\|_{\mathcal{L}(X)} \leq \lambda, \qquad \forall x \in \mathcal{A},$$

  i.e., it holds an (uniformly contraction).

Then for each $\nu \in \left(0, \frac{1}{4} - \lambda\right)$

$$\dim_F(\mathcal{A}; X) \leq \frac{\log N_{(\nu+\lambda)/2\kappa}}{-\log 4(\nu + \lambda)}.$$

A bit about Question 2

Question 2: Among the methods to estimate the fractal dimension of attractors, is it possible to make a comparison (in some sense) between them?

Answer 2: Yes, at least for two of them!

Mañé's method x Smoothing method

- Carvalho, Langa, Robinson (based on Mañé's method, 2010):

$$\dim_F(\mathcal{A}) \leq d \sim N \log N.$$

- Carvalho, Cunha, Langa, Robinson (based on the smoothing for derivatives, 2022):

$$\dim_B(\mathcal{A}) \leq d \sim \log N.$$

Abstract semilinear parabolic problem

$$\begin{cases} u_t + Tu = F(u), & t > 0 \\ u(0) = u_0 \in X^\alpha \end{cases},$$

where

- $T : D(T) \subset X \longrightarrow X$ is a sectorial operator with compact resolvent;
- $F : X^\alpha \longrightarrow X$ is continuously differentiable and Lipschitz on bounded subsets of X^α.

2D-Navier Stokes equation on a periodic domain Q

$$u_t + \mu Tu + B(u,u) = f,$$

where

- $T = -P\Delta$ is the Stokes operator (P is the orthogonal projection onto divergence-free fields);
- $B(u,u) = P[(u \cdot \nabla)u]$ and $\mu > 0$.

In this case,

$$\dim_F(\mathcal{A}; H) \leq cG^4,$$

with

$$G = |f|/\mu^2\lambda_1$$

the Grashof number.

References

For all the details see:

A.N. Carvalho, A.C. Cunha, J.A. Langa & J.C. Robinson, *Finite-dimensional negatively invariant subsets of Banach spaces*. Journal of Mathematical Analysis and Applications **509**, 2022.

For other papers on the fractal dimension see:

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XVIII ENAMA

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