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Refined decay rates of C_0-semigroups on Banach spaces

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Abstract

An important question in the theory of differential equations refers to the asymptotic behavior (in time) of their solutions; more specifically, if they reach an equilibrium and, if so, with which speed. For those linear partial differential equations which can be conveniently analyzed by rewriting them as evolution equations, it is well known that the long-term behavior of the solutions of each one of these equations is related to some spectral properties (and behavior of the resolvent) of the generator of the associated semigroup.

The asymptotic theory of semigroups provides tools for investigating the convergence to zero of mild and classical solutions to the abstract Cauchy problem

$$\begin{cases} u'(t) + Au(t) = 0, & t \geq 0 \\ u(0) = x, \end{cases} \tag{1}$$

We know that (1) has a unique mild solution for every $x \in X$, and that the solution depends continuously on x if, and only if, $-A$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X.

The works of Lebeau [2] (and some others authors) raised the question of what is the relation between the growth rates for norm of the resolvent and the decay rates of the norm of semigroup orbits. More precisely, assuming a spectral condition under the generator, $\sigma(A) \subset \mathbb{C}_+$ in (1), and $\|R(is, A)\|_{\mathcal{L}(X)} \to \infty$ as $|s| \to \infty$, then $(T(t))_{t\geq 0}$ is not exponentially stable and one typically obtains other asymptotic behavior. Until 2010, much attention has been paid to polynomial decay rates of the norm of semigroup orbits. In the work of [**?**], Bátkai, Engel, Prüss and Schnaubelt proved that for uniformly bounded semigroups, a polynomial growth rate of the norm of the resolvent implies a specific polynomial decay rate for classical solutions to (1). more precisely, let $(T(t))_{t\geq 0}$ be a *bounded* semigroup on a Banach space X with infinitesimal generator $-A$ such that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $s \geq 0$ and set $M(s) := \sup_{|\xi| \leq s} \|(i\xi + A)^{-1}\|_{\mathcal{L}(X)}$. If there exist constants $C, \beta > 0$ such that $M(s) \leq C(1+s)^{\beta}$, then for each $\varepsilon > 0$, there exists a positive constant C_ε such that for each $t > 0$, $\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} \leq C_\varepsilon t^{-\frac{1}{\beta}+\varepsilon}$. In [1], Batty and Duyckaerts extended this correspondence to the case where the resolvent growth is arbitrary; they were also able to reduce the loss $\varepsilon > 0$ to a logarithmic scale:

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if $M : (0,\infty) \to (0,\infty)$ is a continuous non-decreasing function of positive increase such that $\|R(is,-A)\|_{\mathcal{L}(X)} \leq M(|s|)$; then, there exists a positive constant C such that $\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O\left(\frac{1}{M_{\log}^{-1}(Ct)}\right)$, $t \to \infty$, where $M_{\log}^{-1}$ is the right inverse of $M_{\log}(s) := M(s)(\log(1+M(s)) + \log(1+s))$. In particular, if $M(s) \leq C(1+s)^{\beta}$ for any $\beta > 0$ and $C > 0$, then $\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O\left(\frac{\log(t)}{t}\right)^{1/\beta}$, $t \to \infty$. Until this point, we have presented some of the main results of the asymptotic theory of **bounded** C_0-semigroups. Nevertheless, there are many natural classes of examples where the norm of the resolvent of the generator grows with a power-law rate as $|s| \to \infty$, for example, but the semigroup is not uniformly bounded, or where it is unknown whether the semigroup is in fact bounded. Using geometrical properties of the underlying Banach space (like its Fourier type), Rozendaal and Veraar have shown the following result (see Theorem 4.9 in [3]). Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator $-A$ defined in a Banach space X with Fourier type $p \in [1,2]$, and let $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Suppose that $\overline{\mathbb{C}_-} \subset \rho(A)$ and that there exist $\beta, C \geq 0$ such that $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C(1+|\lambda|)^{\beta}$ for each $\lambda \in \overline{\mathbb{C}_-}$. Let $\tau > \beta + 1$; then, for each $\rho \in \left[0, \frac{\tau - 1/r}{\beta} - 1\right)$, there exists $C_\rho \geq 0$ such that for each $t \geq 1$, $\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \leq C_\rho t^{-\rho}$.

We have obtained decay rates for C_0-semigroups, by assuming that the norm of the resolvent of the generator behaves as a function of type $|s|^{\beta}\log(|s|)^{b}$ as $|s| \to \infty$ (a particular example of a *regularly varying* function). Under these assumptions on the resolvent and without the assumption of boundedness of the semigroup, to the best knowledge of the authors, these estimates are new. The proofs of the following Theorems can be found in our work [4].

Thoerem: Let $\beta > 0$, $b \geq 0$ and let $(T(t))_{t\geq 0}$ be a C_0-semigroup defined in the Banach space X with Fourier type $p \in [1,2]$, with $-A$ as its generator. Suppose that $\overline{\mathbb{C}_-} \subset \rho(A)$ and that for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \leq 0$, $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim (1+|\lambda|)^{\beta}(\log(2+|\lambda|))^{b}$. Let $r \in [1,\infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$, and let $\tau > 0$ be such that $\tau > \beta + \frac{1}{r}$. Then, for each $\delta > 0$, there exist constants $c_{\delta,\tau} \in [0,\infty)$ and $t_0 \geq 1$ such that for each $t \geq t_0$, $\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \leq c_{\delta,\tau} t^{1-\frac{\tau - r^{-1}}{\beta}} \log(1+t)^{\frac{b(\tau - r^{-1})}{\beta} + \frac{1+\delta}{r}}$.

References

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